Difference between your score and the mean

Suppose you score a 73 on an assignment where the mean is 80.
How do you think of your score?
Do you think I am 7 points below the mean?
What is the expected value of the new derived random variable that is your score minus the mean?

Theorem 3.11

For any random variable $X$,

$$E [X - \mu_X] = 0.$$  

Proof: Theorem 3.11

Defining $g(X) = X - \mu_X$ and applying Theorem 3.10 yields

$$E [g(X)] = \sum_{x \in S_X} (x - \mu_X)P_X(x) = \sum_{x \in S_X} xP_X(x) - \mu_X \sum_{x \in S_X} P_X(x).$$  

The first term on the right side is $\mu_X$ by definition. In the second term, $\sum_{x \in S_X} P_X(x) = 1$, so both terms on the right side are $\mu_X$ and the difference is zero.
**Theorem 3.12**

For any random variable $X$,

$$E[aX + b] = aE[X] + b.$$  

---

**Example 3.30 Problem**

Recall from Examples 3.5 and 3.24 that $X$ has PMF

$$P_X(x) = \begin{cases} 
1/4 & x = 0, \\
1/2 & x = 1, \\
1/4 & x = 2, \\
0 & \text{otherwise.}
\end{cases}$$  

What is the expected value of $V = g(X) = 4X + 7$?

---

**Example 3.30 Solution**

From Theorem 3.12,

$$E[V] = E[g(X)] = E[4X + 7] = 4E[X] + 7 = 4(1) + 7 = 11.$$  

(1)

We can verify this result by applying Theorem 3.10:

$$E[V] = g(0)P_X(0) + g(1)P_X(1) + g(2)P_X(2)$$

$$= 7(1/4) + 11(1/2) + 15(1/4) = 11.$$  

(2)

---

**Example 3.31 Problem**

Continuing Example 3.30, let $W = h(X) = X^2$. What is $E[W]$?

---
Example 3.31 Solution

Theorem 3.10 gives

\[ E[W] = \sum h(x)P_X(x) = (1/4)0^2 + (1/2)1^2 + (1/4)2^2 = 1.5. \]

(1)

Note that this is not the same as \( h(E[W]) = (1)^2 = 1 \).

Quiz 3.7

The number of memory chips \( M \) needed in a personal computer depends on how many application programs, \( A \), the owner wants to run simultaneously. The number of chips \( M \) and the number of application programs \( A \) are described by

\[
M = \begin{cases} 
4 & \text{chips for 1 program,} \\
4 & \text{chips for 2 programs,} \\
6 & \text{chips for 3 programs,} \\
8 & \text{chips for 4 programs,} \\
\end{cases}
\]

\[
P_A(a) = \begin{cases} 
0.1(5-a) & a = 1, 2, 3, 4, \\
0 & \text{otherwise.} \\
\end{cases}
\]

(1)

(a) What is the expected number of programs \( \mu_A = E[A] \)?

(b) Express \( M \), the number of memory chips, as a function \( M = g(A) \) of the number of application programs \( A \).

(c) Find \( E[M] = E[g(A)] \). Does \( E[M] = g(E[A]) \)?

Quiz 3.7 Solution

(a) Using Definition 3.13, the expected number of applications is

\[ E[A] = \sum_{a=1}^{4} aP_A(a) \]

\[ = 1(0.4) + 2(0.3) + 3(0.2) + 4(0.1) \]

\[ = 2. \]

(1)

(b) The number of memory chips is

\[ M = g(A) = \begin{cases} 
4 & A = 1, 2, \\
6 & A = 3, \\
8 & A = 4. \\
\end{cases} \]

(2)

(c) By Theorem 3.10, the expected number of memory chips is

\[ E[M] = \sum_{a=1}^{4} g(A)P_A(a) \]

\[ = 4(0.4) + 4(0.3) + 6(0.2) + 8(0.1) \]

\[ = 4.8. \]

Since \( E[A] = 2 \),

\[ g(E[A]) = g(2) = 4. \]

However, \( E[M] = 4.8 \neq g(E[A]) \). The two quantities are different because \( g(A) \) is not of the form \( \alpha A + \beta \).

Section 3.8

Variance and Standard Deviation
**Definition 3.15 Variance**

The variance of random variable $X$ is

$$\text{Var}[X] = E\left[(X - \mu_X)^2\right].$$

---

**Definition 3.16 Standard Deviation**

The standard deviation of random variable $X$ is

$$\sigma_X = \sqrt{\text{Var}[X]}.$$
**Theorem 3.14**

\[
\text{Var}[X] = E[X^2] - \mu_X^2 = E[X^2] - (E[X])^2.
\]

**Proof: Theorem 3.14**

Expanding the square in (3.75), we have

\[
\begin{align*}
\text{Var}[X] &= \sum_{x \in S_X} x^2 P_X(x) - \sum_{x \in S_X} 2\mu_X x P_X(x) + \sum_{x \in S_X} \mu_X^2 P_X(x) \\
&= E[X^2] - 2\mu_X \sum_{x \in S_X} x P_X(x) + \mu_X^2 \sum_{x \in S_X} P_X(x) \\
&= E[X^2] - 2\mu_X^2 + \mu_X^2.
\end{align*}
\]

**Definition 3.17 Moments**

For random variable \( X \):

(a) The \( n \)th moment is \( E[X^n] \).

(b) The \( n \)th central moment is \( E[(X - \mu_X)^n] \).
Example 3.32 Problem

Continuing Examples 3.5, 3.24, and 3.30, we recall that \( X \) has PMF

\[
P_X(x) = \begin{cases} 
1/4 & x = 0, \\
1/2 & x = 1, \\
1/4 & x = 2, \\
0 & \text{otherwise},
\end{cases}
\]

and expected value \( E[X] = 1 \). What is the variance of \( X \)?

Example 3.32 Solution

In order of increasing simplicity, we present three ways to compute \( \text{Var}[X] \).

- From Definition 3.15, define

\[
W = (X - \mu_X)^2 = (X - 1)^2.
\]

We observe that \( W \equiv 0 \) if and only if \( X = 1 \); otherwise, if \( X = 0 \) or \( X = 2 \), then \( W = 1 \). Thus \( P[W = 0] = P_X(1) = 1/2 \) and \( P[W = 1] = P_X(0) + P_X(2) = 1/2 \). The PMF of \( W \) is

\[
P_W(w) = \begin{cases} 
1/2 & w = 0,1, \\
0 & \text{otherwise}.
\end{cases}
\]

Then

\[
\text{Var}[X] = E[W] = (1/2)(0) + (1/2)(1) = 1/2.
\]

- Recall that Theorem 3.10 produces the same result without requiring the derivation of \( P_W(w) \).

\[
\text{Var}[X] = E[(X - \mu_X)^2] = (0 - 1)^2P_X(0) + (1 - 1)^2P_X(1) + (2 - 1)^2P_X(2) = 1/2.
\]

- To apply Theorem 3.14, we find that

\[
E[X^2] = 0^2P_X(0) + 1^2P_X(1) + 2^2P_X(2) = 1.5.
\]

Thus Theorem 3.14 yields

\[
\text{Var}[X] = E[X^2] - \mu_X^2 = 1.5 - 1^2 = 1/2.
\]

Proof: Theorem 3.15

We let \( Y = aX + b \) and apply Theorem 3.14. We first expand the second moment to obtain

\[
E[Y^2] = E[a^2X^2 + 2abX + b^2] = a^2E[X^2] + 2ab\mu_X + b^2.
\]

Expanding the right side of Theorem 3.12 yields

\[
\mu_Y^2 = a^2\mu_X^2 + 2ab\mu_X + b^2.
\]

Because \( \text{Var}[Y] = E[Y^2] - \mu_Y^2 \), Equations (3.85) and (3.86) imply that

\[
\text{Var}[Y] = a^2E[X^2] - a^2\mu_X^2 = a^2(E[X^2] - \mu_X^2) = a^2 \text{Var}[X].
\]
Example 3.33 Problem

A printer automatically prints an initial cover page that precedes the regular printing of an $X$ page document. Using this printer, the number of printed pages is $Y = X + 1$. Express the expected value and variance of $Y$ as functions of $E[X]$ and $\text{Var}[X]$.

Example 3.33 Solution

The expected number of transmitted pages is $E[Y] = E[X] + 1$. The variance of the number of pages sent is $\text{Var}[Y] = \text{Var}[X]$.

Example 3.34 Problem

In Example 3.28, the amplitude $V$ in volts has PMF

$$P_V(v) = \begin{cases} 1/7 & v = -3, -2, \ldots, 3, \\ 0 & \text{otherwise}. \end{cases}$$

A new voltmeter records the amplitude $U$ in millivolts. Find the variance and standard deviation of $U$.

Example 3.34 Solution

Note that $U = 1000V$. To use Theorem 3.15, we first find the variance of $V$. The expected value of the amplitude is

$$\mu_V = 1/7[-3 + (-2) + (-1) + 0 + 1 + 2 + 3] = 0 \text{ volts}. \quad (1)$$

The second moment is

$$E[V^2] = 1/7[(-3)^2 + (-2)^2 + (-1)^2 + 0^2 + 1^2 + 2^2 + 3^2] = 4 \text{ volts}^2. \quad (2)$$

Therefore the variance is $\text{Var}[V] = E[V^2] - \mu_V^2 = 4 \text{ volts}^2$. By Theorem 3.15,

$$\text{Var}[U] = 1000^2 \text{Var}[V] = 4,000,000 \text{ millivolts}^2,$$

and thus $\sigma_U = 2000$ millivolts.
Theorem 3.16

(a) If \( X \) is Bernoulli \((p)\), then
\[
\text{Var}[X] = p(1-p).
\]

(b) If \( X \) is geometric \((p)\), then
\[
\text{Var}[X] = (1-p)/p^2.
\]

(c) If \( X \) is binomial \((n,p)\), then
\[
\text{Var}[X] = np(1-p).
\]

(d) If \( X \) is Pascal \((k,p)\), then
\[
\text{Var}[X] = k(1-p)/p^2.
\]

(e) If \( X \) is Poisson \((\alpha)\), then
\[
\text{Var}[X] = \alpha.
\]

(f) If \( X \) is discrete uniform \((k,l)\), then
\[
\text{Var}[X] = (l-k)(l-k+2)/12.
\]

Quiz 3.8

In an experiment with three customers entering the Phonesmart store, the observation is \( N \), the number of phones purchased. The PMF of \( N \) is

\[
P_N(n) = \begin{cases} 
\frac{(4-n)}{10} & n = 0, 1, 2, 3 \\
0 & \text{otherwise.}
\end{cases}
\]

Find

(a) The expected value \( \text{E}[N] \)

(b) The second moment \( \text{E}[N^2] \)

(c) The variance \( \text{Var}[N] \)

(d) The standard deviation \( \sigma_N \)

---

Quiz 3.8 Solution

For this problem, it is helpful to wrote out the PMF of \( N \) in the table

\[
\begin{array}{c|cccc}
 n & 0 & 1 & 2 & 3 \\
\hline
 P_N(n) & 0.4 & 0.3 & 0.2 & 0.1 \\
\end{array}
\]

The PMF \( P_N(n) \) allows us to calculate each of the desired quantities.

(a) The expected value is
\[
\text{E}[N] = \sum_{n=0}^{3} nP_N(n) = 0(0.4) + 1(0.3) + 2(0.2) + 3(0.1) = 1. \quad (1)
\]

(b) The second moment of \( N \) is
\[
\text{E}[N^2] = \sum_{n=0}^{3} n^2P_N(n) = 0^2(0.4) + 1^2(0.3) + 2^2(0.2) + 3^2(0.1) = 2. \quad (2)
\]

(c) The variance of \( N \) is
\[
\text{Var}[N] = \text{E}[N^2] - (\text{E}[N])^2 = 2 - 1^2 = 1. \quad (3)
\]

(d) The standard deviation is \( \sigma_N = \sqrt{\text{Var}[N]} = 1. \)