Permutation vs. Combination

Consider the following. EECS offers three scholarships of values $10,000, $5,000, and $1,000. If we have 20 students who have applied for the scholarships, in how many different ways can we award them?

Permutation vs. Combination

When is something a permutation or a combination?

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Example 2.9

- The number of combinations of seven cards chosen from a deck of 52 cards is
  \[
  \binom{52}{7} = \frac{52!}{7!\,45!} = 133,784,560,
  \]
  which is the number of 7-combinations of 52 objects. By contrast, we found in Example 2.5 674,274,182,400 7-permutations of 52 objects. (The ratio is 7! = 5040).

- There are 11 players on a basketball team. The starting lineup consists of five players. There are \( \binom{11}{5} = 462 \) possible starting lineups.

- There are \( \binom{120}{60} \approx 10^{35} \) ways of dividing 120 students enrolled in a probability course into two sections with 60 students in each section.

- A baseball team has 15 field players and ten pitchers. Each field player can take any of the eight nonpitching positions. The starting lineup consists of one pitcher and eight field players. Therefore, the number of possible starting lineups is \( \binom{16}{1}\binom{15}{8} = 64,350 \). For each choice of starting lineup, the manager must submit to the umpire a batting order for the 9 starters. The number of possible batting orders is \( N \times 9! = 23,351,328,000 \) since there are \( N \) ways to choose the 9 starters, and for each choice of 9 starters, there are \( 9! = 362,880 \) possible batting orders.
The book is not correct!

\[
\binom{120}{60} = 9661490884036322603893139521372656
\]

Or

\[
9.6614908840363322603893139521372656 \times 10^{34}
\]
i.e.

96 decillion

Example 2.10 Problem

There are four queens in a deck of 52 cards. You are given seven cards at random from the deck. What is the probability that you have no queens?

Example 2.10 Solution

Consider an experiment in which the procedure is to select seven cards at random from a set of 52 cards and the observation is to determine if there are one or more queens in the selection. The sample space contains \( H = \binom{52}{7} \) possible combinations of seven cards, each with probability \( 1/H \). There are \( H_{NQ} = \binom{52 - 4}{7} \) combinations with no queens. The probability of receiving no queens is the ratio of the number of outcomes with no queens to the number of outcomes in the sample space. \( H_{NQ}/H = 0.5504 \).

Another way of analyzing this experiment is to consider it as a sequence of seven subexperiments. The first subexperiment consists of selecting a card at random and observing whether it is a queen. If it is a queen, an outcome with probability \( 4/52 \) (because there are 52 outcomes in the sample space and four of them are in the event \{queen\}), stop looking for queens.

Example 2.10 Solution (Continued 2)

Otherwise, with probability \( 48/52 \), select another card from the remaining 51 cards and observe whether it is a queen. This outcome of this subexperiment has probability \( 4/51 \). If the second card is not a queen, an outcome with probability \( 47/51 \), continue until you select a queen or you have seven cards with no queen. Using \( Q_i \) and \( N_i \) to indicate a “Queen” or “No queen” on subexperiment \( i \), the tree for this experiment is

\[
\begin{align*}
48/52 & \quad N_1 \\
& \quad 47/51 \quad Q_1 \\
& \quad 46/50 \quad N_2 \quad Q_2 \\
& \quad 45/44 \quad N_3 \quad Q_3 \\
& \quad 44/43 \quad N_4 \quad Q_4 \\
& \quad 43/42 \quad N_5 \quad Q_5 \\
& \quad 42/41 \quad N_6 \quad Q_6 \\
& \quad 41/40 \quad N_7 \quad Q_7
\end{align*}
\]

The probability of the event \( N_T \) that no queen is received in your seven cards is the product of the probabilities of the branches leading to \( N_T \):

\[
(48/52) \times (47/51) \cdots \times (42/46) = 0.5504. \tag{1}
\]
Example 2.11 Problem

There are four queens in a deck of 52 cards. You are given seven cards at random from the deck. After receiving each card you return it to the deck and receive another card at random. Observe whether you have not received any queens among the seven cards you were given. What is the probability that you have received no queens?

Handshakes?

How many handshakes would it take for everyone in the class to shake hands with everyone else? Assume there are 48 students in the class.

Example 2.11 Solution

The sample space contains $52^7$ outcomes. There are $48^7$ outcomes with no queens. The ratio is $(48/52)^7 = 0.5710$, the probability of receiving no queens. If this experiment is considered as a sequence of seven subexperiments, the tree looks the same as the tree in Example 2.10, except that all the horizontal branches have probability $48/52$ and all the diagonal branches have probability $4/52$.

Example 2.12 Problem

A laptop computer has USB slots $A$ and $B$. Each slot can be used for connecting a memory card ($m$), a camera ($c$) or a printer ($p$). It is possible to connect two memory cards, two cameras, or two printers to the laptop. How many ways can we use the two USB slots?
Example 2.12 Solution

This example corresponds to sampling two times with replacement from the set \( \{m,c,p\} \). Let \( xy \) denote the outcome that device type \( x \) is used in slot \( A \) and device type \( y \) is used in slot \( B \). The possible outcomes are \( S = \{mm, mc, mp, cm, cc, cp, pm, pc, pp\} \). The sample space \( S \) contains nine outcomes.

Theorem 2.4

Given \( m \) distinguishable objects, there are \( m^n \) ways to choose with replacement an ordered sample of \( n \) objects.

Example 2.13

There are \( 2^{10} = 1024 \) binary sequences of length 10.

Example 2.14

The letters \( A \) through \( Z \) can produce \( 26^4 = 456,976 \) four-letter words.
Example 2.15

A chip fabrication facility produces microprocessors. Each microprocessor is tested to determine whether it runs reliably at an acceptable clock speed. A subexperiment to test a microprocessor has sample space \( S_{\text{sub}} = \{0, 1\} \) to indicate whether the test was a failure (0) or a success (1). For test \( i \), we record \( x_i = 0 \) or \( x_i = 1 \) to indicate the result. In testing four microprocessors, the observation sequence, \( x_1x_2x_3x_4 \), is one of 16 possible outcomes:

\[
S = \{0000, 0001, 0010, 0011, 0100, 0101, 0110, 0111, 1000, 1001, 1010, 1011, 1100, 1101, 1110, 1111\}.
\]

Example 2.16

There are ten students in a probability class. Each earns a grade \( x \in S_{\text{sub}} = \{A, B, C, F\} \). We use the notation \( x_i \) to denote the grade of the \( i \)th student. For example, the grades for the class could be

\[ x_1x_2\cdots x_{10} = CBBAFBACCF \]  \hspace{1cm} (1)

The sample space \( S \) of possible sequences contains \( 4^{10} = 1,048,576 \) outcomes.

Theorem 2.5

For \( n \) repetitions of a subexperiment with sample space \( S_{\text{sub}} = \{s_0, \ldots, s_{m-1}\} \) the sample space \( S \) of the sequential experiment has \( m^n \) outcomes.

Example 2.17 Problem

For five subexperiments with sample space \( S_{\text{sub}} = \{0, 1\} \), what is the number of observation sequences in which 0 appears \( n_0 = 2 \) times and 1 appears \( n_1 = 3 \) times?
Example 2.17 Solution

The 10 five-letter words with 0 appearing twice and 1 appearing three times are:
{00111, 01011, 01101, 01110, 10011, 10101, 10110, 11001, 11010, 11100}.

Theorem 2.6

The number of observation sequences for $n$ subexperiments with sample space $S = \{0, 1\}$ with 0 appearing $n_0$ times and 1 appearing $n_1 = n - n_0$ times is $\binom{n}{n_1}$.

Theorem 2.7

For $n$ repetitions of a subexperiment with sample space $S = \{s_0, \ldots, s_{m-1}\}$, the number of length $n = n_0 + \cdots + n_{m-1}$ observation sequences with $s_i$ appearing $n_i$ times is

$$\binom{n}{n_0, \ldots, n_{m-1}} = \frac{n!}{n_0! n_1! \cdots n_{m-1}!}.$$ 

Proof: Theorem 2.7

Let $M = \binom{n}{n_0, \ldots, n_{m-1}}$. Start with $n$ empty slots and perform the following sequence of subexperiments:

<table>
<thead>
<tr>
<th>Subexperiment</th>
<th>Procedure</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>Label $n_0$ slots as $s_0$.</td>
</tr>
<tr>
<td>1</td>
<td>Label $n_1$ slots as $s_1$.</td>
</tr>
<tr>
<td>\vdots</td>
<td>\vdots</td>
</tr>
<tr>
<td>$m-1$</td>
<td>Label the remaining $n_{m-1}$ slots as $s_{m-1}$.</td>
</tr>
</tbody>
</table>

There are $\binom{n}{n_0}$ ways to perform subexperiment 0. After $n_0$ slots have been labeled, there are $\binom{n-n_0}{n_1}$ ways to perform subexperiment 1. After subexperiment $j = 1, n_0 + \cdots + n_{j-1}$ slots have already been filled, leaving $\binom{n-(n_0 + \cdots + n_{j-1})}{n_j}$ ways to perform subexperiment $j$. From the fundamental counting principle,

$$M = \binom{n}{n_0} \binom{n-n_0}{n_1} \binom{n-n_0-n_1}{n_2} \cdots \binom{n-n_0-\cdots-n_{m-2}}{n_{m-1}}$$

$$= \frac{n!}{n_0! (n-n_0)!} \frac{1}{n_1! (n-n_0-n_1)!} \cdots \frac{1}{n_{m-1}! (n-n_0-\cdots-n_{m-2})!}$$

$$= \frac{(n-n_0)!n_0! (n-n_0-n_1)!n_1! (n-n_0-\cdots-n_{m-1})!n_{m-1}!}{n!}.$$  \tag{1}

Canceling the common factors, we obtain the formula of the theorem.
Definition 2.2 Multinomial Coefficient

For an integer \( n \geq 0 \), we define

\[
\binom{n}{n_0, \ldots, n_{m-1}} = \begin{cases} 
\frac{n!}{n_0! n_1! \cdots n_{m-1}!} & n_0 + \cdots + n_{m-1} = n, \\
0 & \text{otherwise}. 
\end{cases}
\]

Example 2.18 Problem

In Example 2.16, the professor uses a curve in determining student grades. When there are ten students in a probability class, the professor always issues two grades of A, three grades of B, three grades of C and two grades of F. How many different ways can the professor assign grades to the ten students?

Example 2.18 Solution

In Example 2.16, we determine that with four possible grades there are \( 4^{10} = 1,048,576 \) ways of assigning grades to ten students. However, now we are limited to choosing \( n_0 = 2 \) students to receive an A, \( n_1 = 3 \) students to receive a B, \( n_2 = 3 \) students to receive a C and \( n_3 = 4 \) students to receive an F. The number of ways that fit the curve is the multinomial coefficient

\[
\binom{10}{n_0, n_1, n_2, n_3} = \binom{10}{2,3,3,2} = \frac{10!}{2!3!3!2!} = 25,200. \tag{1}
\]

Quiz 2.2

Consider a binary code with 4 bits (0 or 1) in each code word. An example of a code word is 0110.

(a) How many different code words are there?
(b) How many code words have exactly two zeroes?
(c) How many code words begin with a zero?
(d) In a constant-ratio binary code, each code word has \( N \) bits. In every word, \( M \) of the \( N \) bits are 1 and the other \( N - M \) bits are 0. How many different code words are in the code with \( N = 8 \) and \( M = 3 \)?
Quiz 2.2 Solution

(a) We can view choosing each bit in the code word as a subexperiment. Each subex-
periment has two possible outcomes: 0 and 1. Thus by the fundamental principle of
counting, there are \(2 \times 2 \times 2 \times 2 = 2^4 = 16\) possible code words.

(b) An experiment that can yield all possible code words with two zeroes is to choose
which 2 bits (out of 4 bits) will be zero. The other two bits then must be ones. There
are \(\binom{4}{2} = 6\) ways to do this. Hence, there are six code words with exactly two zeroes.
For this problem, it is also possible to simply enumerate the six code words:

<table>
<thead>
<tr>
<th>1100</th>
<th>1010</th>
<th>1001</th>
</tr>
</thead>
<tbody>
<tr>
<td>0101</td>
<td>0110</td>
<td>0011</td>
</tr>
</tbody>
</table>

(c) When the first bit must be a zero, then the first subexperiment of choosing the
first bit has only one outcome. For each of the next three bits, we have two choices.
In this case, there are \(1 \times 2 \times 2 \times 2 = 8\) ways of choosing a code word.

(d) For the constant ratio code, we can specify a code word by choosing \(M\) of the bits
to be ones. The other \(N - M\) bits will be zeroes. The number of ways of choosing such
a code word is \(\binom{N}{M}\). For \(N = 8\) and \(M = 3\), there are \(\binom{8}{3} = 56\) code words.