Motivation

Suppose we perform a series of actions, each one governed by a probabilistic outcome. The choice of what we do for successive actions might even depend on the previous outcome.

How can we model this process?

One possibility is with a tree diagram.

Example 2.1 Problem

For the resistors of Example 1.19, we used \( A \) to denote the event that a randomly chosen resistor is “within 50 \( \Omega \) of the nominal value.” This could mean “acceptable.” We use the notation \( N \) (“not acceptable”) for the complement of \( A \). The experiment of testing a resistor can be viewed as a two-step procedure. First we identify which machine \( (B_1, B_2, \text{ or } B_3) \) produced the resistor. Second, we find out if the resistor is acceptable. Draw a tree for this sequential experiment. What is the probability of choosing a resistor from machine \( B_2 \) that is not acceptable?
Example 2.1 Solution

This two-step procedure is shown in the tree on the left. To use the tree to find the probability of the event $B_2N$, a nonacceptable resistor from machine $B_2$, we start at the left and find that the probability of reaching $B_2$ is $P[B_2] = 0.4$. We then move to the right to $B_2N$ and multiply $P[B_2]$ by $P[N|B_2] = 0.1$ to obtain $P[B_2N] = (0.4)(0.1) = 0.04$.

Example 2.2 Problem

Traffic engineers have coordinated the timing of two traffic lights to encourage a run of green lights. In particular, the timing was designed so that with probability 0.8 a driver will find the second light to have the same color as the first. Assuming the first light is equally likely to be red or green, what is the probability $P[G_2]$ that the second light is green? Also, what is $P[W]$, the probability that you wait for at least one of the first two lights? Lastly, what is $P[G_1|R_2]$, the conditional probability of a green first light given a red second light?

Example 2.2 Solution

The tree for the two-light experiment is shown on the left. The probability that the second light is green is

$P[G_2] = P[G_1|G_2] + P[R_1|G_2] = 0.4 + 0.1 = 0.5$. \hspace{1cm} (1)

The event $W$ that you wait for at least one light is the event that at least one light is red.

$W = \{R_1G_2 \cup G_1R_2 \cup R_1R_2\}$. \hspace{1cm} (2)

The probability that you wait for at least one light is

$P[W] = P[R_1G_2] + P[G_1R_2] + P[R_1R_2] = 0.1 + 0.1 + 0.4 = 0.6$. \hspace{1cm} (3)

(Continued 2)

An alternative way to the same answer is to observe that $W$ is also the complement of the event that both lights are green. Thus,

$P[W] = P[(G_1G_2)^c] = 1 - P[G_1G_2] = 0.6$. \hspace{1cm} (4)

To find $P[G_1|R_2]$, we need $P[R_2] = 1 - P[G_2] = 0.5$. Since $P[G_1R_2] = 0.1$, the conditional probability that you have a green first light given a red second light is

$P[G_1|R_2] = \frac{P[G_1R_2]}{P[R_2]} = \frac{0.1}{0.5} = 0.2$. \hspace{1cm} (5)

[Continued]
Example 2.3 Problem

Suppose you have two coins, one biased, one fair, but you don’t know which coin is which. Coin 1 is biased. It comes up heads with probability $3/4$, while coin 2 comes up heads with probability $1/2$. Suppose you pick a coin at random and flip it. Let $C_i$ denote the event that coin $i$ is picked. Let $H$ and $T$ denote the possible outcomes of the flip. Given that the outcome of the flip is a head, what is $P[C_1|H]$, the probability that you picked the biased coin? Given that the outcome is a tail, what is the probability $P[C_1|T]$ that you picked the biased coin?

Example 2.3 Solution

First, we construct the sample tree on the left. To find the conditional probabilities, we see

$$P[C_1|H] = \frac{P[C_1|H]}{P[H]} = \frac{P[C_1H]}{P[C_1H] + P[C_2H]}$$

From the leaf probabilities in the sample tree,

$$P[C_1|H] = \frac{3/8}{3/8 + 1/4} = \frac{3}{5}$$

Similarly,

$$P[C_1|T] = \frac{P[C_1T]}{P[T]} = \frac{P[C_1T]}{P[C_1T] + P[C_2T]} = \frac{1/8}{1/8 + 1/4} = \frac{1}{3}$$

As we would expect, we are more likely to have chosen coin 1 when the first flip is heads, but we are more likely to have chosen coin 2 when the first flip is tails.