Given a voltage V, and a resistor of r ohms, what is the power dissipated P?

When designing a circuit, we may know the probability model for V, but need to know the probability model for P. It is not always enough to simply calculate the mean and variance of P.

Modern radio systems, including cell phones often use multiple antennae to receive signals. If the signal amplitudes of each are considered random variables A and B, then what are some ways to optimally combine their signals?

- Selection diversity combining.
- Equal gain combining.
- Maximal ratio combining.

In all these cases, we want to determine properties of the derived random variable W.

How do we do the following?

- Determine $P_W(w)$, the PMF of $W=g(X)$.
- Determine $E[W]$ given $P_X(x)$ and $g(X)$.
- Determine $E[W]$ given $f_X(x)$ and $g(X)$.
- Determine $E[W]$ given $f_{X,Y}(x,y)$ and $g(X,Y)$.
Properties of Derived Random Variables

\[ P_Y(y) = \sum_{x: g(x)=y} P_X(x). \]
\[ E[Y] = \sum_{x \in S_X} g(x) P_X(x). \]
\[ E[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x) \, dx. \]

Discrete: \[ E[W] = \sum_{x \in S_X} \sum_{y \in S_Y} g(x,y) P_{X,Y}(x,y); \]
Continuous: \[ E[W] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x,y) f_{X,Y}(x,y) \, dx \, dy. \]

Section 6.1

PMF of a Function of Two Discrete Random Variables

Basic Approach

First we will want to determine the CDF of \( W \), i.e. \( F_W(w) = P[W \leq w] \) by finding the values of \( X \) for which \( g(X) \leq w \), or if two variables, the values of \( X \) and \( Y \) such that \( g(X,Y) \leq w \).

With this, how can we determine the PDF of \( W \)?

Theorem 6.1

For discrete random variables \( X \) and \( Y \), the derived random variable \( W = g(X,Y) \) has PMF

\[ P_W(w) = \sum_{(x,y): g(x,y)=w} P_{X,Y}(x,y). \]
Example 6.1 Problem

A firm sends out two kinds of newsletters. One kind contains only text and grayscale images and requires 40 cents to print each page. The other kind contains color pictures that cost 60 cents per page. Newsletters can be 1, 2, or 3 pages long. Let the random variable $L$ represent the length of a newsletter in pages. $S_L = \{1, 2, 3\}$. Let the random variable $X$ represent the cost in cents to print each page. $S_X = \{40, 60\}$. After observing many newsletters, the firm has derived the probability model shown above. Let $W = g(L, X) = LX$ be the total cost in cents of a newsletter. Find the range $S_W$ and the PMF $P_W(w)$. 

Example 6.1 Solution

For each of the six possible combinations of $L$ and $X$, we record $W = LX$ under the corresponding entry in the PMF table on the left. The range of $W$ is $S_W = \{40, 60, 80, 120, 180\}$. With the exception of $W = 120$, there is a unique pair $L, X$ such that $W = LX$. For $W = 120$,

$$P_W(120) = P_{L,X}(3, 40) + P_{L,X}(2, 60).$$

The corresponding probabilities are recorded in the second table on the left.

---

Example

Suppose random variables $X$ and $Y$ have the joint PMF:

$$P_{X,Y}(x, y) = \begin{cases} 
|x + y| & x = -2, 0, 2 \\
14 & y = -1, 0, 1 \\
0 & \text{otherwise}
\end{cases}$$

Find the PMF of $W = X - Y$. 

Section 6.2

Functions Yielding Continuous Random Variables
Example 6.2 Problem

In Example 4.2, $W$ centimeters is the location of the pointer on the 1-meter circumference of the circle. Use the solution of Example 4.2 to derive $f_W(w)$.

Example 6.2 Solution

The function $W = 100X$, where $X$ in Example 4.2 is the location of the pointer measured in meters. To find the CDF $F_W(w) = P[W \leq w]$, the first step is to translate the event $\{W \leq w\}$ into an event described by $X$. Each outcome of the experiment is mapped to an $(X, W)$ pair on the line $W = 100X$. Thus the event $\{W \leq w\}$, shown with gray highlight on the vertical axis, is the same event as $\{X \leq w/100\}$, which is shown with gray highlight on the horizontal axis. Both of these events correspond in the figure to observing an $(X, W)$ pair along the highlighted section of the line $w = g(X) = 100X$. This translation of the event $W = w$ to an event described in terms of $X$ depends only on the function $g(X)$. Specifically, it does not depend on the probability model for $X$. From the figure, we see that

$$F_W(w) = P[W \leq w] = P[100X \leq w] = P[X \leq w/100] = F_X(w/100). \quad (1)$$

[Continued]

Example 6.2 Solution (Continued 2)

The calculation of $F_X(w/100)$ depends on the probability model for $X$. For this problem, we recall that Example 4.2 derives the CDF of $X$,

$$F_X(x) = \begin{cases} 0 & x < 0, \\ x & 0 \leq x < 1, \\ 1 & x \geq 1. \end{cases} \quad (1)$$

From this result, we can use algebra to find

$$F_W(w) = F_X\left(\frac{w}{100}\right) = \begin{cases} 0 & \frac{w}{100} < 0, \\ \frac{w}{100} & 0 \leq \frac{w}{100} < 1, \\ 1 & \frac{w}{100} \geq 1, \end{cases} = \begin{cases} 0 & w < 0, \\ \frac{w}{100} & 0 \leq w < 100, \\ 1 & w \geq 100. \end{cases} \quad (2)$$

We take the derivative of the CDF of $W$ over each of the intervals to find the PDF:

$$f_W(w) = \frac{dF_W(w)}{dw} = \begin{cases} 1/100 & 0 \leq w < 100, \\ 0 & \text{otherwise}. \end{cases} \quad (3)$$

We see that $W$ is the uniform $(0, 100)$ random variable.

Theorem 6.2

If $W = aX$, where $a > 0$, then $W$ has CDF and PDF

$$F_W(w) = F_X(w/a), \quad f_W(w) = \frac{1}{a} f_X(w/a).$$
Proof: Theorem 6.2

First, we find the CDF of $W$,

$$F_W(w) = P[aX \leq w] = P[X \leq w/a] = F_X(w/a).$$

(1)

We take the derivative of $F_Y(y)$ to find the PDF:

$$f_W(w) = \frac{dF_W(w)}{dw} = \frac{1}{a} f_X(w/a).$$

(2)

Example 6.3 Problem

The triangular PDF of $X$ is

$$f_X(x) = \begin{cases} 
2x & 0 \leq x \leq 1, \\
0 & \text{otherwise.}
\end{cases}$$

(1)

Find the PDF of $W = aX$. Sketch the PDF of $W$ for $a = 1/2, 1, 2$.

Example 6.3 Solution

For any $a > 0$, we use Theorem 6.2 to find the PDF:

$$f_W(w) = \frac{1}{a} f_X(w/a)$$

$$= \begin{cases} 
2w/a^2 & 0 \leq w \leq a, \\
0 & \text{otherwise.}
\end{cases}$$

(1)

As $a$ increases, the PDF stretches horizontally.

Theorem 6.3

$W = aX$, where $a > 0$.

(a) If $X$ is uniform $(b,c)$, then $W$ is uniform $(ab,ac)$.

(b) If $X$ is exponential $(\lambda)$, then $W$ is exponential $(\lambda/a)$.

(c) If $X$ is Erlang $(n,\lambda)$, then $W$ is Erlang $(n,\lambda/a)$.

(d) If $X$ is Gaussian $(\mu,\sigma)$, then $W$ is Gaussian $(a\mu,a\sigma)$. 
**Theorem 6.4**

If $W = X + b$,

$$F_W(w) = F_X(w - b), \quad f_W(w) = f_X(w - b).$$

---

**Proof: Theorem 6.4**

First, we find the CDF $F_W(w) = P[X + b \leq w] = P[X \leq w - b] = F_X(w - b)$. We take the derivative of $F_W(w)$ to find the PDF: $f_W(w) = dF_W(w)/dw = f_X(w - b)$.

---

**Example 6.4 Problem**

Suppose $X$ is the continuous uniform ($-1, 3$) random variable and $W = X^2$. Find the CDF $F_W(w)$ and PDF $f_W(w)$.

---

**Example 6.4 Solution**

Although $X$ can be negative, $W$ is always non-negative. Thus $F_W(w) = 0$ for $w < 0$. To find the CDF $F_W(w)$ for $w \geq 0$, the figure on the left shows that the event $\{W \leq w\}$, marked with gray highlight on the vertical axis, is the same as the event $\{-\sqrt{w} \leq X \leq \sqrt{w}\}$ marked on the horizontal axis. Both events correspond to $(X, W)$ pairs on the highlighted segment of the function $W = g(X)$. The corresponding algebra is

$$F_W(w) = P[X^2 \leq w] = P[-\sqrt{w} \leq X \leq \sqrt{w}]. \tag{1}$$

[Continued]
Example 6.4 Solution  (Continued 2)

We can take one more step by writing the probability (1) as an integral using the PDF $f_X(x)$:

$$F_W(w) = P \left[-\sqrt{w} \leq X \leq \sqrt{w}\right] = \int_{-\sqrt{w}}^{\sqrt{w}} f_X(x) \, dx. \quad (1)$$

So far, we have used no properties of the PDF $f_X(x)$. However, to evaluate the integral (1), we now recall from the problem statement and Definition 4.5 that the PDF of $X$ is

$$f_X(x) = \begin{cases} 
1/4 & -1 \leq x \leq 3, \\
0 & \text{otherwise.}
\end{cases} \quad (2)$$

The integral (1) is somewhat tricky because the limits depend on the value of $w$. We first observe that $-1 \leq X \leq 3$ implies $0 \leq W \leq 9$. Thus $F_W(w) = 0$ for $w < 0$, and $F_W(w) = 1$ for $w > 9$.

[Continued]

Example 6.4 Solution  (Continued 3)

For $0 \leq w \leq 1$,

$$F_W(w) = \int_{-\sqrt{w}}^{\sqrt{w}} \frac{1}{4} \, dx = \frac{\sqrt{w}}{2}. \quad (1)$$

For $1 \leq w \leq 9$,

$$F_W(w) = \int_{-1}^{\sqrt{w}} \frac{1}{4} \, dx = \frac{\sqrt{w} + 1}{4}. \quad (2)$$

[Continued]

Example 6.4 Solution  (Continued 4)

By combining the separate pieces, we can write a complete expression for $F_W(w)$:

$$F_W(w) = \begin{cases} 
0 & w < 0, \\
\frac{\sqrt{w}}{2} & 0 \leq w \leq 1, \\
\frac{\sqrt{w} + 1}{4} & 1 \leq w \leq 9, \\
1 & w \geq 9.
\end{cases} \quad (1)$$

To find $f_W(w)$, we take the derivative of $F_W(w)$ over each interval.

$$f_W(w) = \begin{cases} 
\frac{1}{4\sqrt{w}} & 0 \leq w \leq 1, \\
\frac{1}{8\sqrt{w}} & 1 \leq w \leq 9, \\
0 & \text{otherwise.}
\end{cases} \quad (2)$$

Example

Let $X$ be a random variable with PDF $f_X$. Find the PDF of the random variable $Y = |X|$ when $f_X(x) = \begin{cases} 1/3, & \text{if } -2 < x \leq 1, \\
0, & \text{otherwise;} \end{cases}$
Solution

Since $Y = |X|$ you can visualize the PDF for any given $y$ as

$$f_Y(y) = \begin{cases} f_X(y) + f_X(-y), & \text{if } y \geq 0, \\ 0, & \text{if } y < 0. \end{cases}$$

Also note that since $Y = |X|$, $Y \geq 0$.

$$f_X(x) = \begin{cases} \frac{1}{3}, & \text{if } -2 < x \leq 1, \\ 0, & \text{otherwise}. \end{cases}$$

So, $f_X(x)$ for $-1 \leq x \leq 0$ gets added to $f_X(x)$ for $0 \leq x \leq 1$:

$$f_Y(y) = \begin{cases} 2/3, & \text{if } 0 \leq y \leq 1, \\ 1/3, & \text{if } 1 < y \leq 2, \\ 0, & \text{otherwise}. \end{cases}$$

---

**Theorem 6.5**

Let $U$ be a uniform $(0,1)$ random variable and let $F(x)$ denote a cumulative distribution function with an inverse $F^{-1}(u)$ defined for $0 < u < 1$. The random variable $X = F^{-1}(U)$ has CDF $F_X(x) = F(x)$.

---

**Example 6.5 Problem**

$U$ is the uniform $(0,1)$ random variable and $X = g(U)$. Derive $g(U)$ such that $X$ is the exponential $(1)$ random variable.
Example 6.5 Solution

The CDF of $X$ is

$$F_X(x) = \begin{cases} 0 & x < 0, \\ 1 - e^{-x} & x \geq 0. \end{cases}$$

(1)

Note that if $u = F_X(x) = 1 - e^{-x}$, then $x = -\ln(1 - u)$. That is, $F_X^{-1}(u) = -\ln(1 - u)$ for $0 \leq u < 1$. Thus, by Theorem 6.5,

$$X = g(U) = -\ln(1 - U)$$

(2)

is the exponential random variable with parameter $\lambda = 1$. Problem 6.2.7 asks the reader to derive the PDF of $X = -\ln(1 - U)$ directly from first principles.

Example 6.6 Problem

For a uniform $(0,1)$ random variable $U$, find a function $g(\cdot)$ such that $X = g(U)$ has a uniform $(a,b)$ distribution.

Example 6.6 Solution

The CDF of $X$ is

$$F_X(x) = \begin{cases} 0 & x < a, \\ (x - a)/(b - a) & a \leq x \leq b, \\ 1 & x > b. \end{cases}$$

(1)

For any $u$ satisfying $0 \leq u \leq 1$, $u = F_X(x) = (x - a)/(b - a)$ if and only if

$$x = F_X^{-1}(u) = a + (b - a)u.$$

(2)

Thus by Theorem 6.5, $X = a + (b - a)U$ is a uniform $(a,b)$ random variable. Note that we could have reached the same conclusion by observing that Theorem 6.3 implies $(b - a)U$ has a uniform $(0,b - a)$ distribution and that Theorem 6.4 implies $a + (b - a)U$ has a uniform $(a,(b - a) + a)$ distribution. Another approach, taken in Problem 6.2.11, is to derive the CDF and PDF of $a + (b - a)U$.

Quiz 6.2

$X$ is an exponential ($\lambda$) PDF. Show that $Y = \sqrt{X}$ is a Rayleigh random variable (see Appendix A.2). Express the Rayleigh parameter $a$ in terms of the exponential parameter $\lambda$. 
Quiz 6.2 Solution

Since \( Y = \sqrt{X} \), the fact that \( X \) is nonnegative implies \( Y \) is non-negative. This implies \( F_Y(y) = 0 \) for \( y < 0 \). For \( y \geq 0 \), we find

\[
F_Y(y) = P(\sqrt{X} \leq y) = P(X \leq y^2) = F_X(y^2).
\]

(1)

For \( x \geq 0 \), \( F_X(x) = 1 - e^{-\lambda x} \). Thus,

\[
F_Y(y) = \begin{cases} 
1 - e^{-\lambda y^2} & y \geq 0 \\
0 & \text{otherwise}
\end{cases}
\]

(2)

By taking the derivative with respect to \( y \), it follows that the PDF of \( Y \) is

\[
f_Y(y) = \begin{cases} 
2\lambda ye^{-\lambda y^2} & y \geq 0 \\
0 & \text{otherwise}
\end{cases}
\]

(3)

In comparing this result to the Rayleigh PDF given in Appendix A, we observe that \( Y \) is a Rayleigh (\( a \)) random variable with \( a = \sqrt{2\lambda} \).