Finally, we note that MATLAB’s random numbers are only seemingly unpredictable. In fact, MATLAB maintains a seed value that determines the subsequent “random” numbers that will be returned. This seed is controlled by the \texttt{rng} function; \texttt{s=rng} saves the current seed and \texttt{rng(s)} restores a previously saved seed. Initializing the random number generator with the same seed always generates the same sequence:

\begin{center}
\begin{tabular}{c|c|c|c|c|c|c|c|c|c}
\texttt{50+randi(50,1,12)} & \texttt{89} & \texttt{76} & \texttt{80} & \texttt{80} & \texttt{72} & \texttt{92} & \texttt{58} & \texttt{56} & \texttt{77} & \texttt{78} & \texttt{59} & \texttt{58} \\
\texttt{50+randi(50,1,12)} & \texttt{89} & \texttt{76} & \texttt{80} & \texttt{80} & \texttt{72} & \texttt{92} & \texttt{58} & \texttt{56} & \texttt{77} & \texttt{78} & \texttt{59} & \texttt{58}
\end{tabular}
\end{center}

When you run a simulation that uses \texttt{rand}, it normally doesn’t matter how the \texttt{rng} seed is initialized. However, it can be instructive to use the same repeatable sequence of \texttt{rand} values when you are debugging your simulation.

\begin{center}
\textbf{Quiz 1.7}
\end{center}

The number of characters in a tweet is equally likely to be any integer between 1 and 140. Simulate an experiment that generates 1000 tweets and counts the number of “long” tweets that have over 120 characters. Repeat this experiment 5 times.

\begin{center}
\textbf{Problems}
\end{center}

<table>
<thead>
<tr>
<th>Difficulty</th>
<th>Easy</th>
<th>Moderate</th>
<th>Difficult</th>
<th>Experts Only</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.1.1</td>
<td></td>
<td></td>
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<tr>
<td>1.1.2</td>
<td></td>
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<tr>
<td>1.1.3</td>
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<tr>
<td>1.2.1</td>
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</table>

\begin{center}
\texttt{Problems}
\end{center}

\begin{center}
\begin{tabular}{l}
Continuing Quiz 1.1, write Gerlanda’s entire menu in words (supply prices if you wish).
\end{tabular}
\end{center}

\begin{center}
\begin{tabular}{l}
For Gerlanda’s pizza in Quiz 1.1, answer these questions:
\end{tabular}
\end{center}

\begin{center}
\begin{tabular}{l}
(a) Are $N$ and $M$ mutually exclusive?
\end{tabular}
\end{center}

\begin{center}
\begin{tabular}{l}
(b) Are $N$, $T$, and $M$ collectively exhaustive?
\end{tabular}
\end{center}

\begin{center}
\begin{tabular}{l}
(c) Are $T$ and $O$ mutually exclusive? State this condition in words.
\end{tabular}
\end{center}

\begin{center}
\begin{tabular}{l}
(d) Does Gerlanda’s make Tuscan pizzas with mushrooms and onions?
\end{tabular}
\end{center}

\begin{center}
\begin{tabular}{l}
(e) Does Gerlanda’s make Neapolitan pizzas that have neither mushrooms nor onions?
\end{tabular}
\end{center}

\begin{center}
\begin{tabular}{l}
Ricardo’s offers customers two kinds of pizza crust, Roman ($R$) and Neapolitan ($N$). All pizzas have cheese but not all pizzas have tomato sauce. Roman pizzas can have tomato sauce or they can be white ($W$); Neapolitan pizzas always have tomato sauce. It is possible to order a Roman pizza with mushrooms ($M$) added. A Neapolitan pizza can contain mushrooms or onions ($O$) or both, in addition to the tomato sauce and cheese. Draw a Venn diagram that shows the relationship among the ingredients $N$, $M$, $O$, $T$, and $W$ in the menu of Ricardo’s pizzeria.
\end{tabular}
\end{center}

\begin{center}
\begin{tabular}{l}
A hypothetical wi-fi transmission can take place at any of three speeds
\end{tabular}
\end{center}
CHAPTER 1  EXPERIMENTS, MODELS, AND PROBABILITIES

depending on the condition of the radio channel between a laptop and an access point. The speeds are high \( h \) at 54 Mb/s, medium \( m \) at 11 Mb/s, and low \( l \) at 1 Mb/s. A user of the wi-fi connection can transmit a short signal corresponding to a mouse click \( c \), or a long signal corresponding to a tweet \( t \). Consider the experiment of monitoring wi-fi signals and observing the transmission speed and the length. An observation is a two-letter word, for example, a high-speed, mouse-click transmission is \( hm \).

(a) What is the sample space of the experiment?
(b) Let \( A_1 \) be the event “medium speed connection.” What are the outcomes in \( A_1 \)?
(c) Let \( A_2 \) be the event “mouse click.” What are the outcomes in \( A_2 \)?
(d) Let \( A_3 \) be the event “high speed connection or low speed connection.” What are the outcomes in \( A_3 \)?
(e) Are \( A_1 \), \( A_2 \), and \( A_3 \) mutually exclusive?
(f) Are \( A_1 \), \( A_2 \), and \( A_3 \) collectively exhaustive?

1.2.2 An integrated circuit factory has three machines \( X \), \( Y \), and \( Z \). Test one integrated circuit produced by each machine. Either a circuit is acceptable \( (a) \) or it fails \( (f) \). An observation is a sequence of three test results corresponding to the circuits from machines \( X \), \( Y \), and \( Z \), respectively. For example, \( aaaf \) is the observation that the circuits from \( X \) and \( Y \) pass the test and the circuit from \( Z \) fails the test.

(a) What are the elements of the sample space of this experiment?
(b) What are the elements of the sets
\[
Z_F = \{ \text{circuit from } Z \text{ fails}\}, \quad X_A = \{ \text{circuit from } X \text{ is acceptable}\}.
\]
(c) Are \( Z_F \) and \( X_A \) mutually exclusive?
(d) Are \( Z_F \) and \( X_A \) collectively exhaustive?
(e) What are the elements of the sets
\[
C = \{ \text{more than one circuit acceptable}\}, \quad D = \{ \text{at least two circuits fail}\}.
\]
(f) Are \( C \) and \( D \) mutually exclusive?
(g) Are \( C \) and \( D \) collectively exhaustive?

1.2.3 Shuffle a deck of cards and turn over the first card. What is the sample space of this experiment? How many outcomes are in the event that the first card is a heart?

1.2.4 Find out the birthday (month and day but not year) of a randomly chosen person. What is the sample space of the experiment? How many outcomes are in the event that the person is born in July?

1.2.5 The sample space of an experiment consists of all undergraduates at a university. Give four examples of partitions.

1.2.6 The sample space of an experiment consists of the measured resistances of two resistors. Give four examples of partitions.

1.3.1 Find \( P[B] \) in each case:
(a) Events \( A \) and \( B \) are a partition and \( P[A] = 3P[B] \).
(b) For events \( A \) and \( B \), \( P[A \cup B] = P[A] \) and \( P[A \cap B] = 0 \).
(c) For events \( A \) and \( B \), \( P[A \cup B] = P[A] - P[B] \).

1.3.2 You roll two fair six-sided dice; one die is red, the other is white. Let \( R_i \) be the event that the red die rolls \( i \). Let \( W_j \) be the event that the white die rolls \( j \).
(a) What is \( P[R_3W_2] \)?
(b) What is the \( P[S_5] \) that the sum of the two rolls is 5?

1.3.3 You roll two fair six-sided dice. Find the probability \( P[D_3] \) that the absolute value of the difference of the dice is 3.

1.3.4 Indicate whether each statement is true or false.
(a) If \( P[A] = 2P[A^c] \), then \( P[A] = 1/2 \).
(b) For all \( A \) and \( B \), \( P[AB] \leq P[A]P[B] \).
(c) If $P[A] < P[B]$, then $P[AB] < P[B]$.
(d) If $P[A \cap B] = P[A]$, then $P[A] \geq P[B]$.

1.3.5 : Computer programs are classified by the length of the source code and by the execution time. Programs with more than 150 lines in the source code are big ($B$). Programs with $\leq 150$ lines are little ($L$). Fast programs ($F$) run in less than 0.1 seconds. Slow programs ($W$) require at least 0.1 seconds. Monitor a program executed by a computer. Observe the length of the source code and the run time. The probability model for this experiment contains the following information: $P[LF] = 0.5$, $P[BF] = 0.2$, and $P[BW] = 0.2$. What is the sample space of the experiment? Calculate the following probabilities: $P[W]$, $P[B]$, and $P[W \cup B]$.

1.3.6 : There are two types of cellular phones, handheld phones ($H$) that you carry and mobile phones ($M$) that are mounted in vehicles. Phone calls can be classified by the traveling speed of the user as fast ($F$) or slow ($W$). Monitor a cellular phone call and observe the type of telephone and the speed of the user. The probability model for this experiment has the following information: $P[F] = 0.5$, $P[HF] = 0.2$, $P[MW] = 0.1$. What is the sample space of the experiment? Find the following probabilities $P[W]$, $P[MF]$, and $P[H]$.

1.3.7 : Shuffle a deck of cards and turn over the first card. What is the probability that the first card is a heart?

1.3.8 : You have a six-sided die that you roll once and observe the number of dots facing upwards. What is the sample space? What is the probability of each sample outcome? What is the probability of $E$, the event that the roll is even?

1.3.9 : A student's score on a 10-point quiz is equally likely to be any integer between 0 and 10. What is the probability of an $A$, which requires the student to get a score of 9 or more? What is the probability the student gets an $F$ by getting less than 4?

1.3.10 : Use Theorem 1.4 to prove the following facts:
(a) $P[A \cup B] \geq P[A]$  
(b) $P[A \cup B] \geq P[B]$  
(c) $P[A \cap B] \leq P[A]$  
(d) $P[A \cap B] \leq P[B]$

1.3.11 : Use Theorem 1.4 to prove by induction the union bound: For any collection of events $A_1, \ldots, A_n$,

$$P[A_1 \cup A_2 \cup \cdots \cup A_n] \leq \sum_{i=1}^{n} P[A_i].$$

1.3.12 : Using only the three axioms of probability, prove $P[\varnothing] = 0$.

1.3.13 : Using the three axioms of probability and the fact that $P[\varnothing] = 0$, prove Theorem 1.3. Hint: Define $A_i = B_i$ for $i = 1, \ldots, m$ and $A_i = \varnothing$ for $i > m$.

1.3.14 : For each fact stated in Theorem 1.4, determine which of the three axioms of probability are needed to prove the fact.

1.4.1 : Mobile telephones perform handoffs as they move from cell to cell. During a call, a telephone either performs zero handoffs ($H_0$), one handoff ($H_1$), or more than one handoff ($H_2$). In addition, each call is either long ($L$), if it lasts more than three minutes, or brief ($B$). The following table describes the probabilities of the possible types of calls.

<table>
<thead>
<tr>
<th></th>
<th>$H_0$</th>
<th>$H_1$</th>
<th>$H_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$L$</td>
<td>0.1</td>
<td>0.1</td>
<td>0.2</td>
</tr>
<tr>
<td>$B$</td>
<td>0.4</td>
<td>0.1</td>
<td>0.1</td>
</tr>
</tbody>
</table>

(a) What is the probability that a brief call will have no handoffs?
(b) What is the probability that a call with one handoff will be long?
(c) What is the probability that a long call will have one or more handoffs?

1.4.2 : You have a six-sided die that you roll once. Let $R_i$ denote the event that the roll is $i$. Let $G_j$ denote the event that
the roll is greater than \( j \). Let \( E \) denote the event that the roll of the die is even-numbered.

(a) What is \( P[R_3|G_1] \), the conditional probability that 3 is rolled given that the roll is greater than 1?

(b) What is the conditional probability that 6 is rolled given that the roll is greater than 3?

(c) What is \( P[G_3|E] \), the conditional probability that the roll is greater than 3 given that the roll is even?

(d) Given that the roll is greater than 3, what is the conditional probability that the roll is even?

1.4.3 You have a shuffled deck of three cards: 2, 3, and 4. You draw one card. Let \( C_i \) denote the event that card \( i \) is picked. Let \( E \) denote the event that the card chosen is an even-numbered card.

(a) What is \( P[C_2|E] \), the probability that the 2 is picked given that an even-numbered card is chosen?

(b) What is the conditional probability that an even-numbered card is picked given that the 2 is picked?

1.4.4 Phonesmart is having a sale on Bananas. If you buy one Banana at full price, you get a second at half price. When couples come in to buy a pair of phones, sales of Apricots and Bananas are equally likely. Moreover, given that the first phone sold is a Banana, the second phone is twice as likely to be a Banana rather than an Apricot. What is the probability that a couple buys a pair of Bananas?

1.4.5 The basic rules of genetics were discovered in mid-1800s by Mendel, who found that each characteristic of a pea plant, such as whether the seeds were green or yellow, is determined by two genes, one from each parent. In his pea plants, Mendel found that yellow seeds were a dominant trait over green seeds. A \( yy \) pea with two yellow genes has yellow seeds; a \( gg \) pea with two recessive genes has green seeds; a hybrid \( gy \) or \( yg \) pea has yellow seeds. In one of Mendel's experiments, he started with a parental generation in which half the pea plants were \( yy \) and half the plants were \( gg \). The two groups were crossbred so that each pea plant in the first generation was \( gy \). In the second generation, each pea plant was equally likely to inherit a \( y \) or a \( g \) gene from each first-generation parent. What is the probability \( P[Y] \) that a randomly chosen pea plant in the second generation has yellow seeds?

1.4.6* From Problem 1.4.5, what is the conditional probability of \( yy \), that a pea plant has two dominant genes given the event \( Y \) that it has yellow seeds?

1.4.7 You have a shuffled deck of three cards: 2, 3, and 4, and you deal out the three cards. Let \( E_i \) denote the event that \( i \)th card dealt is even numbered.

(a) What is \( P[E_2|E_1] \), the probability the second card is even given that the first card is even?

(b) What is the conditional probability that the first two cards are even given that the third card is even?

(c) Let \( O_i \) represent the event that the \( i \)th card dealt is odd numbered. What is \( P[E_2|O_1] \), the conditional probability that the second card is even given that the first card is odd?

(d) What is the conditional probability that the second card is odd given that the first card is odd?

1.4.8 Deer ticks can carry both Lyme disease and human granulocytic ehrlichiosis (HGE). In a study of ticks in the Midwest, it was found that 16% carried Lyme disease, 10% had HGE, and that 10% of the ticks that had either Lyme disease or HGE carried both diseases.

(a) What is the probability \( P[LH] \) that a tick carries both Lyme disease \( (L) \) and HGE \( (H) \)?

(b) What is the conditional probability that a tick has HGE given that it has Lyme disease?
1.5.1 Given the model of handoffs and call lengths in Problem 1.4.1,
(a) What is the probability $P[H_0]$ that a phone makes no handoffs?
(b) What is the probability a call is brief?
(c) What is the probability a call is long or there are at least two handoffs?

1.5.2 For the telephone usage model of Example 1.18, let $B_m$ denote the event that a call is billed for $m$ minutes. To generate a phone bill, observe the duration of the call in integer minutes (rounding up). Charge for $M$ minutes, $M = 1, 2, 3, \ldots$ if the exact duration $T$ is $M - 1 < t \leq M$. A more complete probability model shows that for $m = 1, 2, \ldots$ the probability of each event $B_m$ is

$$P[B_m] = \alpha(1 - \alpha)^{m-1}$$

where $\alpha = 1 - (0.57)^{1/3} = 0.171$.

(a) Classify a call as long, $L$, if the call lasts more than three minutes. What is $P[L]$?
(b) What is the probability that a call will be billed for nine minutes or less?

1.5.3 Suppose a cellular telephone is equally likely to make zero handoffs ($H_0$), one handoff ($H_1$), or more than one handoff ($H_2$). Also, a caller is either on foot ($F$) with probability 5/12 or in a vehicle ($V$).

(a) Given the preceding information, find three ways to fill in the following probability table:

<table>
<thead>
<tr>
<th></th>
<th>$H_0$</th>
<th>$H_1$</th>
<th>$H_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$F$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$V$</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

(b) Suppose we also learn that 1/4 of all callers are on foot making calls with no handoffs and that 1/6 of all callers are vehicle users making calls with a single handoff. Given these additional facts, find all possible ways to fill in the table of probabilities.

1.6.1 Is it possible for $A$ and $B$ to be independent events yet satisfy $A = B$?

1.6.2 Events $A$ and $B$ are equiprobable, mutually exclusive, and independent. What is $P[A]$?

1.6.3 At a Phonesmart store, each phone sold is twice as likely to be an Apricot as a Banana. Also each phone sale is independent of any other phone sale. If you monitor the sale of two phones, what is the probability that the two phones sold are the same?

1.6.4 Use a Venn diagram in which the event areas are proportional to their probabilities to illustrate two events $A$ and $B$ that are independent.

1.6.5 In an experiment, $A$ and $B$ are mutually exclusive events with probabilities $P[A] = 1/4$ and $P[B] = 1/8$.

(a) Find $P[A \cap B]$, $P[A \cup B]$, $P[A \cap B^c]$, and $P[A \cup B^c]$.
(b) Are $A$ and $B$ independent?

1.6.6 In an experiment, $C$ and $D$ are independent events with probabilities $P[C] = 5/8$ and $P[D] = 3/8$.

(a) Determine the probabilities $P[C \cap D]$, $P[C \cap D^c]$, and $P[C^c \cap D^c]$.
(b) Are $C^c$ and $D^c$ independent?

1.6.7 In an experiment, $A$ and $B$ are mutually exclusive events with probabilities $P[A \cup B] = 5/8$ and $P[A] = 3/8$.

(a) Find $P[B]$, $P[A \cap B^c]$, and $P[A \cup B^c]$.
(b) Are $A$ and $B$ independent?

1.6.8 In an experiment, $C$, and $D$ are independent events with probabilities $P[C \cap D] = 1/3$, and $P[C] = 1/2$.

(a) Find $P[D]$, $P[C \cap D^c]$, and $P[C^c \cap D^c]$.
(b) Find $P[C \cup D]$ and $P[C \cup D^c]$.
(c) Are $C$ and $D^c$ independent?

1.6.9 In an experiment with equiprobable outcomes, the sample space is $S = \{1, 2, 3, 4\}$ and $P[s] = 1/4$ for all $s \in S$. Find three events in $S$ that are pairwise independent but are not independent. (Note:
Pairwise independent events meet the first three conditions of Definition 1.7).

1.6.10 (Continuation of Problem 1.4.5) One of Mendel’s most significant results was the conclusion that genes determining different characteristics are transmitted independently. In pea plants, Mendel found that round peas \((r)\) are a dominant trait over wrinkled peas \((w)\). Mendel crossbred a group of \((rr, yy)\) peas with a group of \((ww, gg)\) peas. In this notation, \(rr\) denotes a pea with two “round” genes and \(ww\) denotes a pea with two “wrinkled” genes. The first generation were either \((rw, yg)\), \((rw, gy)\), \((wr, yg)\), or \((wr, gy)\) plants with both hybrid shape and hybrid color. Breeding among the first generation yielded second-generation plants in which genes for each characteristic were equally likely to be either dominant or recessive. What is the probability \(P[Y]\) that a second-generation pea plant has yellow seeds? What is the probability \(P[R]\) that a second-generation plant has round peas? Are \(R\) and \(Y\) independent events? How many visibly different kinds of pea plants would Mendel observe in the second generation? What are the probabilities of each of these kinds?

1.6.11 For independent events \(A\) and \(B\), prove that
(a) \(A\) and \(B^c\) are independent.
(b) \(A^c\) and \(B\) are independent.
(c) \(A^c\) and \(B^c\) are independent.

1.6.12 Use a Venn diagram in which the event areas are proportional to their probabilities to illustrate three events \(A\), \(B\), and \(C\) that are independent.

1.6.13 Use a Venn diagram in which event areas are in proportion to their probabilities to illustrate events \(A\), \(B\), and \(C\) that are pairwise independent but not independent.

1.7.1 Following Quiz 1.3, use MATLAB, but not the \texttt{randi} function, to generate a vector \(T\) of 200 independent test scores such that all scores between 51 and 100 are equally likely.
scalar $X_i$ or a vector or matrix $X$. In addition, we use $X_{i,j}$ to denote the $i,j$th element. Thus, $X$ and $x$ (in a MATLAB code fragment) may both refer to the same variable.

**Quiz 2.5**

The flip of a thick coin yields heads with probability 0.4, tails with probability 0.5, or lands on its edge with probability 0.1. Simulate 100 thick coin flips. Your output should be a $3 \times 1$ vector $X$ such that $X_1$, $X_2$, and $X_3$ are the number of occurrences of heads, tails, and edge.

**Problems**

<table>
<thead>
<tr>
<th>Difficulty</th>
<th>Easy</th>
<th>Moderate</th>
<th>Difficult</th>
<th>Experts Only</th>
</tr>
</thead>
</table>

2.1.1 Suppose you flip a coin twice. On any flip, the coin comes up heads with probability $1/4$. Use $H_i$ and $T_i$ to denote the result of flip $i$.

(a) What is the probability, $P[H_1|H_2]$, that the first flip is heads given that the second flip is heads?

(b) What is the probability that the first flip is heads and the second flip is tails?


2.1.3 At the end of regulation time, a basketball team is trailing by one point and a player goes to the line for two free throws. If the player makes exactly one free throw, the game goes into overtime. The probability that the first free throw is good is $1/2$. However, if the first attempt is good, the player relaxes and the second attempt is good with probability $3/4$. However, if the player misses the first attempt, the added pressure reduces the success probability to $1/4$. What is the probability that the game goes into overtime?

2.1.4 You have two biased coins. Coin $A$ comes up heads with probability $1/4$. Coin $B$ comes up heads with probability $3/4$. However, you are not sure which is which, so you choose a coin randomly and you flip it. If the flip is heads, you guess that the flipped coin is $B$; otherwise, you guess that the flipped coin is $A$. What is the probability $P[C]$ that your guess is correct?

2.1.5 Suppose that for the general population, 1 in 5000 people carries the human immunodeficiency virus (HIV). A test for the presence of HIV yields either a positive (+) or negative (−) response. Suppose the test gives the correct answer 99% of the time. What is $P[\neg H]$, the conditional probability that a person tests negative given that the person does not have the HIV virus? What is $P[H+|\neg H]$, the conditional probability that a randomly chosen person has the HIV virus given that the person tests positive?

2.1.6 A machine produces photo detectors in pairs. Tests show that the first photo detector is acceptable with probability $3/5$. When the first photo detector is acceptable, the second photo detector is acceptable with probability $4/5$. If the first photo detector is defective, the second photo detector is acceptable with probability $2/5$.

(a) Find the probability that exactly one photo detector of a pair is acceptable.

(b) Find the probability that both photo detectors in a pair are defective.

2.1.7 You have two biased coins. Coin $A$ comes up heads with probability $1/4$. Coin $B$ comes up heads with probability $3/4$. However, you are not sure which is which so you flip each coin once, choosing the first coin randomly. Use $H_i$ and $T_i$ to denote the result of flip $i$. Let $A_1$ be the event that coin $A$ was flipped first. Let $B_1$ be the event that coin $B$ was flipped first. What is $P[H_1H_2]$?
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Are \( H_1 \) and \( H_2 \) independent? Explain your answer.

2.1.8 A particular birth defect of the heart is rare; a newborn infant will have the defect \( D \) with probability \( P[D] = 10^{-4} \). In the general exam of a newborn, a particular heart arrhythmia \( A \) occurs with probability 0.99 in infants with the defect. However, the arrhythmia also appears with probability 0.1 in infants without the defect. When the arrhythmia is present, a lab test for the defect is performed. The result of the lab test is either positive (event \( T^+ \)) or negative (event \( T^- \)). In a newborn with the defect, the lab test is positive with probability \( p = 0.999 \) independent from test to test. In a newborn without the defect, the lab test is negative with probability \( p = 0.999 \). If the arrhythmia is present and the test is positive, then heart surgery (event \( H \)) is performed.

(a) Given the arrhythmia \( A \) is present, what is the probability the infant has the defect \( D \)?

(b) Given that an infant has the defect, what is the probability \( P[H|D] \) that heart surgery is performed?

(c) Given that the infant does not have the defect, what is the probability \( q = P[H|D^c] \) that an unnecessary heart surgery is performed?

(d) Find the probability \( P[H] \) that an infant has heart surgery performed for the arrhythmia.

(e) Given that heart surgery is performed, what is the probability that the newborn does not have the defect?

2.1.9 Suppose Dagwood (Blondie’s husband) wants to eat a sandwich but needs to go on a diet. Dagwood decides to let the flip of a coin determine whether he eats. Using an unbiased coin, Dagwood will postpone the diet (go directly to the refrigerator) if either (a) he flips heads on his first flip or (b) he flips tails on the first flip but then proceeds to get two heads out of the next three flips. Note that the first flip is not counted in the attempt to win two of three and that Dagwood never performs any unnecessary flips. Let \( H_1 \) be the event that Dagwood flips heads on try i. Let \( T_i \) be the event that tails occurs on flip i.

(a) Draw the tree for this experiment. Label the probabilities of all outcomes.

(b) What are \( P[H_3] \) and \( P[T_3] \)?

(c) Let \( D \) be the event that Dagwood must diet. What is \( P[D] \)? What is \( P[H_1|D] \)?

(d) Are \( H_3 \) and \( H_2 \) independent events?

2.1.10 The quality of each pair of photo detectors produced by the machine is independent of the quality of every other pair of detectors.

(a) What is the probability of finding no good detectors in a collection of \( n \) pairs produced by the machine?

(b) How many pairs of detectors must the machine produce to reach a probability of 0.99 that there will be at least one acceptable photo detector?

2.1.11 In Steven Strogatz’s New York Times blog http://opinionator.blogs.nytimes.com/2010/04/25/chances-are/?ref=opinion, the following problem was posed to highlight the confusing character of conditional probabilities.

Before going on vacation for a week, you ask your spacey friend to water your ailing plant. Without water, the plant has a 90 percent chance of dying. Even with proper watering, it has a 20 percent chance of dying. And the probability that your friend will forget to water it is 30 percent. (a) What’s the chance that your plant will survive the week? (b) If it’s dead when you return, what’s the chance that your friend forgot to water it? (c) If your friend forgot to water it, what’s the chance it’ll be dead when you return?

Solve parts (a), (b) and (c) of this problem.

2.1.12 Each time a fisherman casts his line, a fish is caught with probability \( p \), independent of whether a fish is caught on any other cast of the line. The fisherman will fish all day until a fish is caught and
then he will quit and go home. Let $C_i$ denote the event that on cast $i$, the fisherman catches a fish. Draw the tree for this experiment and find $P[C_1], P[C_2],$ and $P[C_n]$ as functions of $p$.

2.2.1 * On each turn of the knob, a gumball machine is equally likely to dispense a red, yellow, green or blue gumball, independent from turn to turn. After eight turns, what is the probability $P[R_y Y_2 G_2 B_2]$ that you have received 2 red, 2 yellow, 2 green and 2 blue gumballs?

2.2.2 * A Starburst candy package contains 12 individual candy pieces. Each piece is equally likely to be berry, orange, lemon, or cherry, independent of all other pieces.
(a) What is the probability that a Starburst package has only berry or cherry pieces and zero orange or lemon pieces?
(b) What is the probability that a Starburst package has no cherry pieces?
(c) What is the probability $P[F_1]$ of all twelve pieces of your Starburst are the same flavor?

2.2.3 * Your Starburst candy has 12 pieces, three pieces of each of four flavors: berry, lemon, orange, and cherry, arranged in a random order in the pack. You draw the first three pieces from the pack.
(a) What is the probability they are all the same flavor?
(b) What is the probability they are all different flavors?

2.2.4 * Your Starburst candy has 12 pieces, three pieces of each of four flavors: berry, lemon, orange, and cherry, arranged in a random order in the pack. You draw the first four pieces from the pack.
(a) What is the probability $P[F_1]$ they are all the same flavor?
(b) What is the probability $P[F_4]$ they are all different flavors?
(c) What is the probability $P[F_2]$ that your Starburst has exactly two pieces of each of two different flavors?

2.2.5 * In a game of rummy, you are dealt a seven-card hand.
(a) What is the probability $P[R_7]$ that your hand has only red cards?
(b) What is the probability $P[F]$ that your hand has only face cards?
(c) What is the probability $P[R_7 F]$ that your hand has only red face cards? (The face cards are jack, queen, and king.)

2.2.6 * In a game of poker, you are dealt a five-card hand.
(a) What is the probability $P[R_5]$ that your hand has only red cards?
(b) What is the probability of a "full house" with three-of-a-kind and two-of-a-kind?

2.2.7 * Consider a binary code with 5 bits (0 or 1) in each code word. An example of a code word is 01010. How many different code words are there? How many code words have exactly three 0's?

2.2.8 * Consider a language containing four letters: A, B, C, D. How many three-letter words can you form in this language? How many four-letter words can you form if each letter appears only once in each word?

2.2.9 * On an American League baseball team with 15 field players and 10 pitchers, the manager selects a starting lineup with 8 field players, 1 pitcher, and 1 designated hitter. The lineup specifies the players for these positions and the positions in a batting order for the 8 field players and designated hitter. If the designated hitter must be chosen among all the field players, how many possible starting lineups are there?

2.2.10 * Suppose that in Problem 2.2.9, the designated hitter can be chosen from among all the players. How many possible starting lineups are there?

2.2.11 * At a casino, the only game is a numberless roulette. On a spin of the wheel, the ball lands in a space with color red (r), green (g), or black (b). The wheel has 19 red spaces, 19 green spaces and 2 black spaces.
(a) In 40 spins of the wheel, find the probability of the event

\[ A = \{19 \text{ reds, 19 greens, and 2 blacks}\}. \]

(b) In 40 spins of the wheel, find the probability of \( G_{19} = \{19 \text{ greens}\}. \)

(c) The only bets allowed are red and green. Given that you randomly choose to bet red or green, what is the probability \( p \) that your bet is a winner?

2.2.12 A basketball team has three pure centers, four pure forwards, four pure guards, and one swingman who can play either guard or forward. A pure position player can play only the designated position. If the coach must start a lineup with one center, two forwards, and two guards, how many possible lineups can the coach choose?

2.2.13 An instant lottery ticket consists of a collection of boxes covered with gray wax. For a subset of the boxes, the gray wax hides a special mark. If a player scratches off the correct number of the marked boxes (and no boxes without the mark), then that ticket is a winner. Design an instant lottery game in which a player scratches five boxes and the probability that a ticket is a winner is approximately 0.01.

2.3.1 Consider a binary code with 5 bits (0 or 1) in each code word. An example of a code word is 01010. In each code word, a bit is a zero with probability 0.8, independent of any other bit.

(a) What is the probability of the code word 00111?

(b) What is the probability that a code word contains exactly three ones?

2.3.2 The Boston Celtics have won 16 NBA championships over approximately 50 years. Thus it may seem reasonable to assume that in a given year the Celtics win the title with probability \( p = 16/50 = 0.32 \), independent of any other year. Given such a model, what would be the probability of the Celtics winning eight straight championships beginning in 1959? Also, what would be the probability of the Celtics winning the title in 10 out of 11 years, starting in 1959? Given your answers, do you trust this simple probability model?

2.3.3 Suppose each day that you drive to work a traffic light that you encounter is either green with probability 7/16, red with probability 7/16, or yellow with probability 1/8, independent of the status of the light on any other day. If over the course of five days, \( G, Y, \) and \( R \) denote the number of times the light is found to be green, yellow, or red, respectively, what is the probability that \( P[G = 2, Y = 1, R = 2] \)? Also, what is the probability \( P[G = R] \)?

2.3.4 In a game between two equal teams, the home team wins with probability \( p > 1/2 \). In a best of three playoff series, a team with the home advantage has a game at home, followed by a game away, followed by a home game if necessary. The series is over as soon as one team wins two games. What is \( P[H] \), the probability that the team with the home advantage wins the series? Is the home advantage increased by playing a three-game series rather than a one-game playoff? That is, is it true that \( P[H] \geq p \) for all \( p \geq 1/2 \)?

2.3.5 A collection of field goal kickers are divided into groups 1 and 2. Group i has 3i kickers. On any kick, a kicker from group i will kick a field goal with probability \( 1/(i+1) \), independent of the outcome of any other kicks.

(a) A kicker is selected at random from among all the kickers and attempts one field goal. Let \( K \) be the event that a field goal is kicked. Find \( P[K] \).

(b) Two kickers are selected at random; \( K_j \) is the event that kicker \( j \) kicks a field goal. Are \( K_1 \) and \( K_2 \) independent?

(c) A kicker is selected at random and attempts 10 field goals. Let \( M \) be the number of misses. Find \( P[M = 5] \).
2.4.1 A particular operation has six components. Each component has a failure probability \( q \), independent of any other component. A successful operation requires both of the following conditions:

- Components 1, 2, and 3 all work, or component 4 works.
- Component 5 or component 6 works.

Draw a block diagram for this operation similar to those of Figure 2.2 on page 53. Derive a formula for the probability \( P[W] \) that the operation is successful.

2.4.2 We wish to modify the cellular telephone coding system in Example 2.21 in order to reduce the number of errors. In particular, if there are two or three zeroes in the received sequence of 5 bits, we will say that a deletion (event \( D \)) occurs. Otherwise, if at least 4 zeroes are received, the receiver decides a zero was sent, or if at least 4 ones are received, the receiver decides a one was sent. We say that an error occurs if \( i \) was sent and the receiver decides \( j \neq i \) was sent. For this modified protocol, what is the probability \( P[E] \) of an error? What is the probability \( P[D] \) of a deletion?

2.4.3 Suppose a 10-digit phone number is transmitted by a cellular phone using four binary symbols for each digit, using the model of binary symbol errors and deletions given in Problem 2.4.2. Let \( C \) denote the number of bits sent correctly, \( D \) the number of deletions, and \( E \) the number of errors. Find \( P[C = c, D = d, E = e] \) for all \( c, d, \) and \( e \).

2.4.4 Consider the device in Problem 2.4.1. Suppose we can replace any one component with an ultrareliable component that has a failure probability of \( q/2 = 0.05 \). Which component should we replace?

2.5.1 Build a MATLAB simulation of 50 trials of the experiment of Example 2.3. Your output should be a pair of \( 50 \times 1 \) vectors \( C \) and \( H \). For the \( i \)th trial, \( H_i \) will record whether it was heads \( (H_i = 1) \) or tails \( (H_i = 0) \), and \( C_i \in \{1, 2\} \) will record which coin was picked.

2.5.2 Following Quiz 2.3, suppose the communication link has different error probabilities for transmitting 0 and 1. When a 1 is sent, it is received as a 0 with probability 0.01. When a 0 is sent, it is received as a 1 with probability 0.03. Each bit in a packet is still equally likely to be a 0 or 1. Packets have been coded such that if five or fewer bits are received in error, then the packet can be decoded. Simulate the transmission of 100 packets, each containing 100 bits. Count the number of packets decoded correctly.

2.5.3 For a failure probability \( q = 0.2 \), simulate 100 trials of the six-component test of Problem 2.4.1. How many devices were found to work? Perform 10 repetitions of the 100 trials. What do you learn from 10 repetitions of 100 trials compared to a simulated experiment with 100 trials?

2.5.4 Write a MATLAB function

\[ N \text{countequal}(G, T) \]

that duplicates the action of \( \text{hist}(G, T) \) in Example 2.26. Hint: Use \( \text{ndgrid} \).

2.5.5 In this problem, we use a MATLAB simulation to "solve" Problem 2.4.4. Recall that a particular operation has six components. Each component has a failure probability \( q \) independent of any other component. The operation is successful if both

- Components 1, 2, and 3 all work, or component 4 works.
- Component 5 or component 6 works.

With \( q = 0.2 \), simulate the replacement of a component with an ultrareliable component. For each replacement of a regular component, perform 100 trials. Are 100 trials sufficient to decide which component should be replaced?
of samples \( x(1), x(2), \ldots, x(n) \) of a random variable \( X \) will converge to \( \mathbb{E}[X] \) as \( n \) becomes large. For a discrete uniform \((0, 10)\) random variable \( X \), use MATLAB to examine this convergence.

(a) For 100 sample values of \( X \), plot the sequence \( m_1, m_2, \ldots, m_{100} \). Repeat this experiment five times, plotting all five \( m_n \) curves on common axes.
(b) Repeat part (a) for 1000 sample values of \( X \).

### Problems

#### 3.2.1 ★ The random variable \( N \) has PMF

\[
P_N(n) = \begin{cases} 
  c(1/2)^n & n = 0, 1, 2, \\
  0 & \text{otherwise}
\end{cases}
\]

(a) What is the value of the constant \( c \)?
(b) What is \( P[N \leq 1] \)?

#### 3.2.2 ★ The random variable \( V \) has PMF

\[
P_V(v) = \begin{cases} 
  cv^2 & v = 1, 2, 3, 4, \\
  0 & \text{otherwise}
\end{cases}
\]

(a) Find the value of the constant \( c \).
(b) Find \( P[V \in \{u^2 | u = 1, 2, 3, \ldots\}] \).
(c) Find the probability that \( V \) is even.
(d) Find \( P[V > 2] \).

#### 3.2.3 ★ The random variable \( X \) has PMF

\[
P_X(x) = \begin{cases} 
  c/x & x = 2, 4, 8, \\
  0 & \text{otherwise}
\end{cases}
\]

(a) What is the value of the constant \( c \)?
(b) What is \( P[X = 4] \)?
(c) What is \( P[X < 4] \)?
(d) What is \( P[3 \leq X \leq 9] \)?

#### 3.2.4 ★ In each at-bat in a baseball game, mighty Casey swings at every pitch. The

result is either a home run (with probability \( q = 0.05 \)) or a strike. Of course, three strikes and Casey is out.

(a) What is the probability \( P[H] \) that Casey hits a home run?
(b) For one at-bat, what is the PMF of \( N \), the number of times Casey swings his bat?

#### 3.2.5 ★ A tablet computer transmits a file over a wi-fi link to an access point. Depending on the size of the file, it is transmitted as \( N \) packets where \( N \) has PMF

\[
P_N(n) = \begin{cases} 
  c/n & n = 1, 2, 3, \\
  0 & \text{otherwise}
\end{cases}
\]

(a) Find the constant \( c \).
(b) What is the probability that \( N \) is odd?
(c) Each packet is received correctly with probability \( p \), and the file is received correctly if all \( N \) packets are received correctly. Find \( P[C] \), the probability that the file is received correctly.

#### 3.2.6 ★ In college basketball, when a player is fouled while not in the act of shooting and the opposing team is “in the penalty,” the player is awarded a “1 and 1.” In the 1 and 1, the player is awarded one free throw, and if that free throw goes in the player is awarded a second free throw. Find the PMF of \( Y \), the number of points scored in
a 1 and 1 given that any free throw goes in with probability \( p \), independent of any other free throw.

**3.2.7** You roll a 6-sided die repeatedly. Starting with roll \( i = 1 \), let \( R_i \) denote the result of roll \( i \). If \( R_i > i \), then you will roll again; otherwise you stop. Let \( N \) denote the number of rolls.

(a) What is \( P[N > 3] \)?
(b) Find the PMF of \( N \).

**3.2.8** You are manager of a ticket agency that sells concert tickets. You assume that people will call three times in an attempt to buy tickets and then give up. You want to make sure that you are able to serve at least 95% of the people who want tickets. Let \( p \) be the probability that a caller gets through to your ticket agency. What is the minimum value of \( p \) necessary to meet your goal?

**3.2.9** In the ticket agency of Problem 3.2.8, each telephone ticket agent is available to receive a call with probability 0.2. If all agents are busy when someone calls, the caller hears a busy signal. What is the minimum number of agents that you have to hire to meet your goal of serving 95% of the customers who want tickets?

**3.2.10** Suppose when a baseball player gets a hit, a single is twice as likely as a double, which is twice as likely as a triple, which is twice as likely as a home run. Also, the player's batting average, i.e., the probability the player gets a hit, is 0.300. Let \( B \) denote the number of bases touched safely during an at-bat. For example, \( B = 0 \) when the player makes an out, \( B = 1 \) on a single, and so on. What is the PMF of \( B \)?

**3.2.11** When someone presses SEND on a cellular phone, the phone attempts to set up a call by transmitting a SETUP message to a nearby base station. The phone waits for a response, and if none arrives within 0.5 seconds it tries again. If it doesn’t get a response after \( n = 6 \) tries, the phone stops transmitting messages and generates a busy signal.

(a) Draw a tree diagram that describes the call setup procedure.
(b) If all transmissions are independent and the probability is \( p \) that a SETUP message will get through, what is the PMF of \( K \), the number of messages transmitted in a call attempt?
(c) What is the probability that the phone will generate a busy signal?
(d) As manager of a cellular phone system, you want the probability of a busy signal to be less than 0.02. If \( p = 0.9 \), what is the minimum value of \( n \) necessary to achieve your goal?

**3.3.1** In a package of M&Ms, \( Y \), the number of yellow M&Ms, is uniformly distributed between 5 and 15.

(a) What is the PMF of \( Y \)?
(b) What is \( P[Y < 10] \)?
(c) What is \( P[Y > 12] \)?
(d) What is \( P[8 \leq Y \leq 12] \)?

**3.3.2** In a bag of 25 M&Ms, each piece is equally likely to be red, green, orange, blue, or brown, independent of the color of any other piece. Find the PMF of \( R \), the number of red pieces. What is the probability a bag has no red M&Ms?

**3.3.3** When a conventional paging system transmits a message, the probability that the message will be received by the pager it is sent to is \( p \). To be confident that a message is received at least once, a system transmits the message \( n \) times.

(a) Assuming all transmissions are independent, what is the PMF of \( K \), the number of times the pager receives the same message?
(b) Assume \( p = 0.8 \). What is the minimum value of \( n \) that produces a probability of 0.95 of receiving the message at least once?

**3.3.4** You roll a pair of fair dice until you roll “doubles” (i.e., both dice are the same). What is the expected number, \( E[N] \), of rolls?
3.3.5 When you go fishing, you attach \( m \) hooks to your line. Each time you cast your line, each hook will be swallowed by a fish with probability \( h \), independent of whether any other hook is swallowed. What is the PMF of \( K \), the number of fish that are hooked on a single cast of the line?

3.3.6 Any time a child throws a Frisbee, the child’s dog catches the Frisbee with probability \( p \), independent of whether the Frisbee is caught on any previous throw. When the dog catches the Frisbee, it runs away with the Frisbee, never to be seen again. The child continues to throw the Frisbee until the dog catches it. Let \( X \) denote the number of times the Frisbee is thrown.

(a) What is the PMF \( P_X(x) \)?

(b) If \( p = 0.2 \), what is the probability that the child will throw the Frisbee more than four times?

3.3.7 When a two-way paging system transmits a message, the probability that the message will be received by the pager it is sent to is \( p \). When the pager receives the message, it transmits an acknowledgment signal (ACK) to the paging system. If the paging system does not receive the ACK, it sends the message again.

(a) What is the PMF of \( N \), the number of times the system sends the same message?

(b) The paging company wants to limit the number of times it has to send the same message. It has a goal of \( P[N \leq 3] \geq 0.95 \). What is the minimum value of \( p \) necessary to achieve the goal?

3.3.8 The number of bytes \( B \) in an HTML file is the geometric \((2.5 \cdot 10^{-5})\) random variable. What is the probability \( P[B > 500,000] \) that a file has over 500,000 bytes?

3.3.9

(a) Starting on day 1, you buy one lottery ticket each day. Each ticket is a winner with probability 0.1. Find the PMF of \( K \), the number of tickets you buy up to and including your fifth winning ticket.

(b) \( L \) is the number of flips of a fair coin up to and including the 33rd occurrence of tails. What is the PMF of \( L \)?

(c) Starting on day 1, you buy one lottery ticket each day. Each ticket is a winner with probability 0.01. Let \( M \) equal the number of tickets you buy up to and including your first winning ticket. What is the PMF of \( M \)?

3.3.10 The number of buses that arrive at a bus stop in \( T \) minutes is a Poisson random variable \( B \) with expected value \( T/5 \).

(a) What is the PMF of \( B \), the number of buses that arrive in \( T \) minutes?

(b) What is the probability that in a two-minute interval, three buses will arrive?

(c) What is the probability of no buses arriving in a 10-minute interval?

(d) How much time should you allow so that with probability 0.99 at least one bus arrives?

3.3.11 In a wireless automatic meter-reading system, a base station sends out a wake-up signal to nearby electric meters. On hearing the wake-up signal, a meter transmits a message indicating the electric usage. Each message is repeated eight times.

(a) If a single transmission of a message is successful with probability \( p \), what is the PMF of \( N \), the number of successful message transmissions?

(b) \( I \) is an indicator random variable such that \( I = 1 \) if at least one message is transmitted successfully; otherwise \( I = 0 \). Find the PMF of \( I \).

3.3.12 A Zipf \((n, \alpha = 1)\) random variable \( X \) has PMF

\[
P_X(x) = \begin{cases} 
    c(n)/x & x = 1, 2, \ldots, n, \\
    0 & \text{otherwise.}
\end{cases}
\]

The constant \( c(n) \) is set so that \( \sum_{x=1}^{n} P_X(x) = 1 \). Calculate \( c(n) \) for \( n = 1, 2, \ldots, 6 \).
3.3.13 In a bag of 64 “holiday season” M&Ms, each M&M is equally likely to be red or green, independent of any other M&M in the bag.

(a) If you randomly grab four M&Ms, what is the probability $P[E]$ that you grab an equal number of red and green M&Ms?

(b) What is the PMF of $G$, the number of green M&Ms in the bag?

(c) You begin eating randomly chosen M&Ms one by one. Let $R$ equal the number of red M&Ms you eat before you eat your first green M&M. What is the PMF of $R$?

3.3.14 A radio station gives a pair of concert tickets to the sixth caller who knows the birthday of the performer. For each person who calls, the probability is 0.75 of knowing the performer’s birthday. All calls are independent.

(a) What is the PMF of $L$, the number of calls necessary to find the winner?

(b) What is the probability of finding the winner on the tenth call?

(c) What is the probability that the station will need nine or more calls to find a winner?

3.3.15 In a packet voice communications system, a source transmits packets containing digitized speech to a receiver. Because transmission errors occasionally occur, an acknowledgment (ACK) or a negative acknowledgment (NAK) is transmitted back to the source to indicate the status of each received packet. When the transmitter gets a NAK, the packet is retransmitted. Voice packets are delay sensitive, and a packet can be transmitted a maximum of $d$ times. If a packet transmission is an independent Bernoulli trial with success probability $p$, what is the PMF of $T$, the number of times a packet is transmitted?

3.3.16 At Newark airport, your jet joins a line as the tenth jet waiting for takeoff. At Newark, takeoffs and landings are synchronized to the minute. In each one-minute interval, an arriving jet lands with probability $p = 2/3$, independent of an arriving jet in any other minute. Such an arriving jet blocks any waiting jet from taking off in that one-minute interval. However, if there is no arrival, then the waiting jet at the head of the line takes off. Each take-off requires exactly one minute.

(a) Let $L_i$ denote the number of jets that land before the jet at the front of the line takes off. Find the PMF $P_{L_i}(l)$.

(b) Let $W$ denote the number of minutes you wait until your jet takes off. Find $P[W = 10]$. (Note that if no jets land for ten minutes, then one waiting jet will take off each minute and $W = 10$.)

(c) What is the PMF of $W$?

3.3.17 Suppose each day (starting on day 1) you buy one lottery ticket with probability $1/2$; otherwise, you buy no tickets. A ticket is a winner with probability $p$ independent of the outcome of all other tickets. Let $N_i$ be the event that on day $i$ you do not buy a ticket. Let $W_i$ be the event that on day $i$, you buy a winning ticket. Let $L_i$ be the event that on day $i$ you buy a losing ticket.

(a) What are $P[W_{34}]$, $P[L_{87}]$, and $P[N_{50}]$?

(b) Let $K$ be the number of the day on which you buy your first lottery ticket. Find the PMF $P_K(k)$.

(c) Find the PMF of $R$, the number of losing lottery tickets you have purchased in $m$ days.

(d) Let $D$ be the number of the day on which you buy your $j$th losing ticket. What is $P_D(d)$? Hint: If you buy your $j$th losing ticket on day $d$, how many losers did you have after $d - 1$ days?

3.3.18 The Sixers and the Celtics play a best out of five playoff series. The series ends as soon as one of the teams has won three games. Assume that each team is equally likely to win any game independently of any other game played. Find

(a) The PMF $P_N(n)$ for the total number $N$ of games played in the series;
3.3.19 For a binomial random variable $K$ representing the number of successes in $n$ trials, $\sum_{k=0}^{n} P_K(k) = 1$. Use this fact to prove the binomial theorem for any $a > 0$ and $b > 0$. That is, show that

$$(a + b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}.$$ 

3.4.1 Discrete random variable $Y$ has the CDF $F_Y(y)$ as shown:

<table>
<thead>
<tr>
<th>$F_Y(y)$</th>
<th>$0.75$</th>
<th>$0.5$</th>
<th>$0.25$</th>
<th>$0$</th>
<th>$1$</th>
<th>$1$</th>
<th>$2$</th>
<th>$3$</th>
<th>$4$</th>
<th>$5$</th>
<th>$y$</th>
</tr>
</thead>
</table>

Use the CDF to find the following probabilities:

(a) $P[Y < 1]$ and $P[Y \leq 1]$
(b) $P[Y > 2]$ and $P[Y \geq 2]$
(c) $P[Y = 3]$ and $P[Y > 3]$
(d) $P_Y(y)$

3.4.2 The random variable $X$ has CDF $F_X(x)$ as follows:

$$F_X(x) = \begin{cases} 
0 & x < -1, \\
0.2 & -1 \leq x < 0, \\
0.7 & 0 \leq x < 1, \\
1 & x \geq 1.
\end{cases}$$

(a) Draw a graph of the CDF.
(b) Write $P_X(x)$, the PMF of $X$.

3.4.4 Following Example 3.22, show that a geometric $(p)$ random variable $K$ has CDF

$$F_K(k) = \begin{cases} 
0 & k < 1, \\
1 - (1 - p)^{k+1} & k \geq 1.
\end{cases}$$

3.4.5 At the One Top Pizza Shop, a pizza sold has mushrooms with probability $p = 2/3$. On a day in which 100 pizzas are sold, let $N$ equal the number of pizzas sold before the first pizza with mushrooms is sold. What is the PMF of $N$? What is the CDF of $N$?

3.4.6 In Problem 3.2.10, find and sketch the CDF of $B$, the number of bases touched safely during an at-bat.

3.4.7 In Problem 3.2.6, find and sketch the CDF of $Y$, the number of points scored in a 1 and 1 for $p = 1/4$, $p = 1/2$, and $p = 3/4$.

3.4.8 In Problem 3.2.11, find and sketch the CDF of $N$, the number of attempts made by the cellular phone for $p = 1/2$.

3.5.1 Let $X$ have the uniform PMF

$$P_X(x) = \begin{cases} 
0.01 & x = 1, 2, \ldots, 100, \\
0 & \text{otherwise}.
\end{cases}$$

(a) Find a mode $x_{\text{mod}}$ of $X$. If the mode is not unique, find the set $X_{\text{mod}}$ of all modes of $X$.
(b) Find a median $x_{\text{med}}$ of $X$. If the median is not unique, find the set $X_{\text{med}}$ of all numbers $x$ that are medians of $X$.

3.5.2 It costs 20 cents to receive a photo and 30 cents to send a photo from a cell-phone. $C$ is the cost of one photo (either sent or received). The probability of receiving a photo is 0.6. The probability sending a photo is 0.4.

(a) Find $P_C(c)$, the PMF of $C$.
(b) What is $E[C]$, the expected value of $C$?

3.5.3  
(a) The number of trains $J$ that arrive at the station in time $t$ minutes is a Poisson random variable with $E[J] = t$. Find $t$ such that $P[J > 0] = 0.9$.

(b) The number of buses $K$ that arrive at the station in one hour is a Poisson random variable with $E[K] = 10$. Find $P[K = 10]$.

(c) In a 1 ms interval, the number of hits $L$ on a Web server is a Poisson random variable with expected value $E[L] = 2$ hits. What is $P[L \leq 1]$?

3.5.4  
You simultaneously flip a pair of fair coins. Your friend gives you one dollar if both coins come up heads. You repeat this ten times and your friend gives you $X$ dollars. Find $E[X]$, the expected number of dollars you receive. What is the probability that you do “worse than average”?

3.5.5  
A packet received by your smartphone is error-free with probability 0.95, independent of any other packet.

(a) Out of 10 packets received, let $X$ equal the number of packets received with errors. What is the PMF of $X$?

(b) In one hour, your smartphone receives 12,000 packets. Let $X$ equal the number of packets received with errors. What is $E[X]$?

3.5.6  
Find the expected value of the random variable $Y$ in Problem 3.4.1.

3.5.7  
Find the expected value of the random variable $X$ in Problem 3.4.2.

3.5.8  
Find the expected value of the random variable $X$ in Problem 3.4.3.

3.5.9  
Use Definition 3.13 to calculate the expected value of a binomial $(4, 1/2)$ random variable $X$.

3.5.10  
$X$ is the discrete uniform $(1, 5)$ random variable.

(a) What is $P[X = E[X]]$?

(b) What is $P[X > E[X]]$?

3.5.11  
$K$ is the geometric $(1/11)$ random variable.

(a) What is $P[K = E[K]]$?

(b) What is $P[K > E[K]]$?

(c) What is $P[K < E[K]]$?

3.5.12  
At a casino, people line up to pay $20 each to be a contestant in the following game: The contestant flips a fair coin repeatedly. If she flips heads 20 times in a row, she walks away with $R = 20$ million dollars; otherwise she walks away with $R = 0$ dollars.

(a) Find the PMF of $R$, the reward earned by the contestant.

(b) The casino counts “losing contestants” who fail to win the 20 million dollar prize. Let $L$ equal the number of losing contestants before the first winning contestant. What is the PMF of $L$?

(c) Why does the casino offer this game?

3.5.13  
Give examples of practical applications of probability theory that can be modeled by the following PMFs. In each case, state an experiment, the sample space, the range of the random variable, the PMF of the random variable, and the expected value:

(a) Bernoulli

(b) Binomial

(c) Pascal

(d) Poisson

Make up your own examples. (Don’t copy examples from the text.)

3.5.14  
Find $P[K < E[K]]$ when

(a) $K$ is geometric $(1/3)$.

(b) $K$ is binomial $(6, 1/2)$.

(c) $K$ is Poisson $(3)$.

(d) $K$ is discrete uniform $(0, 6)$.

3.5.15  
Suppose you go to a casino with exactly $63. At this casino, the only game is roulette and the only bets allowed are red and green. The payoff for a winning bet
is the amount of the bet. In addition, the wheel is fair so that \( P[\text{red}] = P[\text{green}] = 1/2 \). You have the following strategy: First, you bet $1. If you win the bet, you quit and leave the casino with $64. If you lose, you then bet $2. If you win, you quit and go home. If you lose, you bet $4. In fact, whenever you lose, you double your bet until either you win a bet or you lose all of your money. However, as soon as you win a bet, you quit and go home. Let \( Y \) equal the amount of money that you take home. Find \( P_Y(y) \) and \( E[Y] \). Would you like to play this game every day?

3.5.16 In a TV game show, there are three identical-looking suitcases. The first suitcase has 3 dollars, the second has 30 dollars and the third has 300 dollars. You start the game by randomly choosing a suitcase. Between the two unchosen suitcases, the game show host opens the suitcase with more money. The host then asks you if you want to keep your suitcase or switch to the other remaining suitcase. After you make your decision, you open your suitcase and keep the \( D \) dollars inside. Should you switch suitcases? To answer this question, solve the following subproblems and use the following notation:

- \( C_i \) is the event that you first choose the suitcase with \( i \) dollars.
- \( O_i \) denotes the event that the host opens a suitcase with \( i \) dollars.

In addition, you may wish to go back and review the Monty Hall problem in Example 2.4.

(a) Suppose you never switch; you always stick with your original choice. Use a tree diagram to find the PMF \( P_D(d) \) and expected value \( E[D] \).

(b) Suppose you always switch. Use a tree diagram to find the PMF \( P_D(d) \) and expected value \( E[D] \).

(c) Perhaps your rule for switching should depend on how many dollars are in the suitcase that the host opens? What is the optimal strategy to maximize \( E[D] \)? Hint: Consider making a random decision: if the host opens a suitcase with \( i \) dollars, let \( \alpha_i \) denote the probability that you switch.

3.5.17 You are a contestant on a TV game show; there are four identical-looking suitcases containing $100, $200, $400, and $800. You start the game by randomly choosing a suitcase. Among the three unchosen suitcases, the game show host opens the suitcase that holds the median amount of money. (For example, if the unopened suitcases contain $100, $400 and $800, the host opens the $400 suitcase.) The host then asks you if you want to keep your suitcase or switch one of the other remaining suitcases. For your analysis, use the following notation for events:

- \( C_i \) is the event that you choose a suitcase with \( i \) dollars.
- \( O_i \) denotes the event that the host opens a suitcase with \( i \) dollars.
- \( R \) is the reward in dollars that you keep.

(a) You refuse the host’s offer and open the suitcase you first chose. Find the PMF of \( R \) and the expected value \( E[R] \).

(b) You always switch and randomly choose one of the two remaining suitcases with equal probability. You receive the \( R \) dollars in this chosen suitcase. Sketch a tree diagram for this experiment, and find the PMF and expected value of \( R \).

(c) Can you do better than either always switching or always staying with your original choice? Explain.

3.5.18 You are a contestant on a TV game show; there are four identical-looking suitcases containing $200, $400, $800, and $1600. You start the game by randomly choosing a suitcase. Among the three unchosen suitcases, the game show host opens the suitcase that holds the least money. The host then asks you if you want to keep
your suitcase or switch one of the other remaining suitcases. For the following analysis, use the following notation for events:

- $C_i$ is the event that you choose a suitcase with $i$ dollars.
- $O_i$ denotes the event that the host opens a suitcase with $i$ dollars.
- $R$ is the reward in dollars that you keep.

(a) You refuse the host’s offer and open the suitcase you first chose. Find the PMF of $R$ and the expected value $E[R]$.

(b) You switch and randomly choose one of the two remaining suitcases. You receive the $R$ dollars in this chosen suitcase. Sketch a tree diagram for this experiment, and find the PMF and expected value of $R$.

3.5.19 Let binomial random variable $X_n$ denote the number of successes in $n$ Bernoulli trials with success probability $p$. Prove that $E[X_n] = np$. Hint: Use the fact that $\sum_{x=0}^{n-1} P_{X_n}(x) = 1$.

3.5.20 Prove that if $X$ is a nonnegative integer-valued random variable, then

$$E[X] = \sum_{k=0}^{\infty} P[X > k].$$

3.5.21 At the gym, a weightlifter can bench press a maximum of 100 kg. For a mass of $m$ kg, ($1 \leq m \leq 100$), the maximum number of repetitions she can complete is $R$, a geometric random variable with expected value $100/m$.

(a) In terms of the mass $m$, what is the PMF of $R$?

(b) When she performs one repetition, she lifts the $m$ kg mass a height $h = 1.98$ meters and thus does work $w = mgh = 4m$ Joules. For $R$ repetitions, she does $W = 4mR$ Joules of work. What is the expected work $E[W]$ that she will complete?

(c) A friend offers to pay her 1000 dollars if she can perform 1000 Joules of weightlifting work. What mass $m$ in the range $1 \leq m \leq 200$ should she use to maximize her probability of winning the money?

3.5.22 At the gym, a weightlifter can bench press a maximum of 200 kg. For a mass of $m$ kg, ($1 \leq m \leq 200$), the maximum number of repetitions she can complete is $R$, a geometric random variable with expected value $200/m$.

(a) In terms of the mass $m$, what is the PMF of $R$?

(b) When she performs one repetition, she lifts the $m$ kg mass a height $h = 1.98$ meters and thus does work $w = mgh = 4m$ Joules. For $R$ repetitions, she does $W = 4mR$ Joules of work. What is the expected work $E[W]$ that she will complete?

(c) A friend offers to pay her 1000 dollars if she can perform 1000 Joules of weightlifting work. What mass $m$ in the range $1 \leq m \leq 200$ should she use to maximize her probability of winning the money?

3.6.1 Given the random variable $Y$ in Problem 3.4.1, let $U = g(Y) = Y^2$.

(a) Find $P_Y(y)$.

(b) Find $F_Y(y)$.

(c) Find $E[U]$.

3.6.2 Given the random variable $X$ in Problem 3.4.2, let $V = g(X) = |X|$.

(a) Find $P_V(v)$.

(b) Find $F_V(v)$.

(c) Find $E[V]$.

3.6.3 Given the random variable $X$ in Problem 3.4.3, let $W = g(X) = -X$.

(a) Find $P_W(w)$.

(b) Find $F_W(w)$.

(c) Find $E[W]$.
3.6.4 Auf a discount brokerage, a stock purchase or sale worth less than $10,000 incurs a brokerage fee of 1% of the value of the transaction. A transaction worth more than $10,000 incurs a fee of $100 plus 0.5% of the amount exceeding $10,000. Note that for a fraction of a cent, the brokerage always charges the customer a full penny. You wish to buy 100 shares of a stock whose price $D$ in dollars has PMF

$$P_D(d) = \begin{cases} 1/3 & d = 99.75, 100, 100.25, \\ 0 & \text{otherwise.} \end{cases}$$

What is the PMF of $C$, the cost of buying the stock (including the brokerage fee)?

3.6.5 A source transmits data packets to a receiver over a radio link. The receiver uses error detection to identify packets that have been corrupted by radio noise. When a packet is received error free, the receiver sends an acknowledgment (ACK) back to the source. When the receiver gets a packet with errors, a negative acknowledgment (NAK) message is sent back to the source. Each time the source receives a NAK, the packet is retransmitted. We assume that each packet transmission is independently corrupted by errors with probability $q$.

(a) Find the PMF of $X$, the number of times that a packet is transmitted by the source.

(b) Suppose each packet takes 1 millisecond to transmit and that the source waits an additional millisecond to receive the acknowledgment message (ACK or NAK) before retransmitting. Let $T$ equal the time required until the packet is successfully received. What is the relationship between $T$ and $X$? What is the PMF of $T$?

3.6.6 Suppose that a cellular phone costs $20 per month with 30 minutes of use included and that each additional minute of use costs $0.50. If the number of minutes you use in a month is a geometric random variable $M$ with expected value of $E[M] = 1/p = 30$ minutes, what is the PMF of $C$, the cost of the phone for one month?

3.6.7 A professor tries to count the number of students attending lecture. For each student in the audience, the professor either counts the student properly (with probability $p$) or overlooks (and does not count) the student with probability $1 - p$. The exact number of attending students is 70.

(a) The number of students counted by the professor is a random variable $N$. What is the PMF of $N$?

(b) Let $U = 70 - N$ denote the number of uncounted students. What is the PMF of $U$?

(c) What is the probability that the undercount $U$ is 2 or more?

(d) For what value of $p$ does $E[U] = 2$?

3.6.8 A forgetful professor tries to count the M&Ms in a package; however, the professor often loses his place and double counts an M&M. For each M&M in the package, the professor counts the M&M and then, with probability $p$ counts the M&M again. The exact number of M&Ms in the pack is 20.

(a) Find the PMF of $R$, the number of double-counted M&Ms.

(b) Find the PMF of $N$, the number of M&Ms counted by the professor.

3.7.1 Starting on day $n = 1$, you buy one lottery ticket each day. Each ticket costs 1 dollar and is independently a winner that can be cashed for 5 dollars with probability 0.1; otherwise the ticket is worthless. Let $X_n$ equal your net profit after $n$ days. What is $E[X_n]$?

3.7.2 For random variable $T$ in Quiz 3.6, first find the expected value $E[T]$ using Theorem 3.10. Next, find $E[T]$ using Definition 3.13.

3.7.3 In a certain lottery game, the chance of getting a winning ticket is exactly one in a thousand. Suppose a person buys one ticket each day (except on the leap year day February 29) over a period of fifty years.
What is the expected number $E[T]$ of winning tickets in fifty years? If each winning ticket is worth $1000, what is the expected amount $E[R]$ collected on these winning tickets? Lastly, if each ticket costs $2, what is your expected net profit $E[Q]$?

**3.7.4** Suppose an NBA basketball player shooting an uncontested 2-point shot will make the basket with probability 0.6. However, if you foul the shooter, the shot will be missed, but two free throws will be awarded. Each free throw is an independent Bernoulli trial with success probability $p$. Based on the expected number of points the shooter will score, for what values of $p$ may it be desirable to foul the shooter?

**3.7.5** It can take up to four days after you call for service to get your computer repaired. The computer company charges for repairs according to how long you have to wait. The number of days $D$ until the service technician arrives and the service charge $C$, in dollars, are described by

<table>
<thead>
<tr>
<th>$d$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P_D(d)$</td>
<td>0.2</td>
<td>0.4</td>
<td>0.3</td>
<td>0.1</td>
</tr>
</tbody>
</table>

and

$$C = \begin{cases} 
90 & \text{for 1-day service,} \\
70 & \text{for 2-day service,} \\
40 & \text{for 3-day service,} \\
40 & \text{for 4-day service.} 
\end{cases}$$

(a) What is the expected waiting time $\mu_D = E[D]$?

(b) What is the expected deviation $E[D - \mu_D]$?

(c) Express $C$ as a function of $D$.

(d) What is the expected value $E[C]$?

**3.7.6** True or False: For any random variable $X$, $E[1/X] = 1/E[X]$.

**3.7.7** For the cellular phone in Problem 3.6.6, express the monthly cost $C$ as a function of $M$, the number of minutes used. What is the expected monthly cost $E[C]$?

**3.7.8** A new cellular phone billing plan costs $15 per month plus $1 for each minute of use. If the number of minutes you use the phone in a month is a geometric random variable with expected value $1/p$, what is the expected monthly cost $E[C]$ of the phone? For what values of $p$ is this billing plan preferable to the billing plan of Problem 3.6.6 and Problem 3.7.7?

**3.7.9** A particular circuit works if all 10 of its component devices work. Each circuit is tested before leaving the factory. Each working circuit can be sold for $k$ dollars, but each nonworking circuit is worthless and must be thrown away. Each circuit can be built with either ordinary devices or ultra-reliable devices. An ordinary device has a failure probability of $q = 0.1$ and costs $1$. An ultra-reliable device has a failure probability of $q/2$ but costs $3$. Assuming device failures are independent, should you build your circuit with ordinary devices or ultra-reliable devices in order to maximize your expected profit $E[R]$? Keep in mind that your answer will depend on $k$.

**3.7.10** In the New Jersey state lottery, each $1 ticket has six randomly marked numbers out of 1, \ldots, 46. A ticket is a winner if the six marked numbers match six numbers drawn at random at the end of a week. For each ticket sold, 50 cents is added to the pot for the winners. If there are $k$ winning tickets, the pot is divided equally among the $k$ winners. Suppose you bought a winning ticket in a week in which $2n$ tickets are sold and the pot is $n$ dollars.

(a) What is the probability $q$ that a random ticket will be a winner?

(b) Find the PMF of $K_n$, the number of other (besides your own) winning tickets.

(c) What is the expected value of $W_n$, the prize for your winning ticket?

**3.7.11** If there is no winner for the lottery described in Problem 3.7.10, then the pot is carried over to the next week. Suppose that in a given week, an $r$ dollar pot is carried over from the previous week and
2n tickets sold. Answer the following questions.

(a) What is the probability \( q \) that a random ticket will be a winner?

(b) If you own one of the 2n tickets sold, what is the expected value of \( V \), the value (i.e., the amount you win) of that ticket? Is it ever possible that \( E[V] > 1? \)

(c) Suppose that in the instant before the ticket sales are stopped, you are given the opportunity to buy one of each possible ticket. For what values (if any) of \( n \) and \( r \) should you do it?

3.8.1 In an experiment to monitor two packets, the PMF of \( N \), the number of video packets, is

\[
<table>
<thead>
<tr>
<th>n</th>
<th>P_N(n)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.2</td>
</tr>
<tr>
<td>1</td>
<td>0.7</td>
</tr>
<tr>
<td>2</td>
<td>0.1</td>
</tr>
</tbody>
</table>
\]

Find \( E[N], E[N^2], \) Var[\( N \)], and \( \sigma_N. \)

3.8.2 Find the variance of the random variable \( Y \) in Problem 3.4.1.

3.8.3 Find the variance of the random variable \( X \) in Problem 3.4.2.

3.8.4 Find the variance of the random variable \( X \) in Problem 3.4.3.

3.8.5 Let \( X \) have the binomial PMF

\[
P_X(x) = \binom{4}{x} \left(\frac{1}{2}\right)^x.
\]

(a) Find the standard deviation of \( X \).

(b) What is \( \Pr[\mu_X - \sigma_X \leq X \leq \mu_X + \sigma_X] \), the probability that \( X \) is within one standard deviation of the expected value?

3.8.6 \( X \) is the binomial (5, 0.5) random variable.

(a) Find the standard deviation of \( X \).

(b) Find \( \Pr[\mu_X - \sigma_X \leq X \leq \mu_X + \sigma_X] \), the probability that \( X \) is within one standard deviation of the expected value.

3.8.7 Show that the variance of \( Y = aX + b \) is \( \text{Var}[Y] = a^2 \text{Var}[X] \).

3.8.8 Given a random variable \( X \) with expected value \( \mu_X \) and variance \( \sigma_X^2 \), find the expected value and variance of

\[
Y = \frac{X - \mu_X}{\sigma_X}.
\]

3.8.9 In real-time packet data transmission, the time between successfully received packets is called the interarrival time, and randomness in packet interarrival times is called jitter. Jitter is undesirable. One measure of jitter is the standard deviation of the packet interarrival time. From Problem 3.6.5, calculate the jitter \( \sigma_T \). How large must the successful transmission probability \( q \) be to ensure that the jitter is less than 2 milliseconds?

3.8.10 Random variable \( K \) has a Poisson (\( \alpha \)) distribution. Derive the properties \( E[K] = \text{Var}[K] = \alpha \). Hint: \( E[K^2] = E[K(K - 1)] + E[K] \).

3.8.11 For the delay \( D \) in Problem 3.7.5, what is the standard deviation \( \sigma_D \) of the waiting time?

3.9.1 Let \( X \) be the binomial (100, 0.1/2) random variable. Let \( E_2 \) denote the event that \( X \) is a perfect square. Calculate \( \Pr[E_2] \).

3.9.2 Write a MATLAB function \( \text{shipcostpmf.m} \) that produces \( m \) random sample values of the package weight \( X \) with PMF given in Example 3.27.

3.9.3 Use the \texttt{unique} function to write a MATLAB script \( \text{shipcostpmf.m} \) that outputs the pair of vectors \( \text{sy} \) and \( \text{py} \) representing the PMF \( P_Y(y) \) of the shipping cost \( Y \) in Example 3.27.

3.9.4 For \( m = 10, m = 100, \) and \( m = 1000 \), use MATLAB to find the average cost of sending \( m \) packages using the model of Example 3.27. Your program input should have the number of trials \( m \) as the input. The output should be \( \bar{Y} = \frac{1}{m} \sum_{i=1}^{m} Y_i \), where \( Y_i \) is the cost of the \( i \)th package. As \( m \) becomes large, what do you observe?
3.9.5 The Zipf \((n, \alpha = 1)\) random variable \(X\) introduced in Problem 3.3.12 is often used to model the "popularity" of a collection of \(n\) objects. For example, a Web server can deliver one of \(n\) Web pages. The pages are numbered such that the page 1 is the most requested page, page 2 is the second most requested page, and so on. If page \(k\) is requested, then \(X = k\).

To reduce external network traffic, an ISP gateway caches copies of the \(k\) most popular pages. Calculate, as a function of \(n\) for \(1 \leq n \leq 1000\), how large \(k\) must be to ensure that the cache can deliver a page with probability 0.75.

3.9.6 Generate \(n\) independent samples of the Poisson \((\lambda)\) random variable \(Y\). For each \(y \in S_Y\), let \(n(y)\) denote the number of times that \(y\) was observed. Thus \(\sum_{y \in S_Y} n(y) = n\) and the relative frequency of \(y\) is \(R(y) = n(y)/n\). Compare the relative frequency of \(y\) against \(P_Y(y)\) by plotting \(R(y)\) and \(P_Y(y)\) on the same graph as functions of \(y\) for \(n = 100\), \(n = 1000\) and \(n = 10,000\). How large should \(n\) be to have reasonable agreement?

3.9.7 Test the convergence of Theorem 3.8. For \(\alpha = 10\), plot the PMF of \(K_n\) for \((n, p) = (10, 1), (n, p) = (100, 0.1),\) and \((n, p) = (1000, 0.01)\) and compare each result with the Poisson (\(\alpha\)) PMF.

3.9.8 Use the result of Problem 3.4.4 and the Random Sample Algorithm on Page 102 to write a MATLAB function \(k = \text{geometricrv}(p, m)\) that generates \(m\) samples of a geometric (\(p\)) random variable.

3.9.9 Find \(n^*\), the smallest value of \(n\) for which the function \(\text{poissonpmf}(n, n)\) shown in Example 3.37 reports an error. What is the source of the error? Write a function \(\text{bigpoissonpmf}(\alpha, n)\) that calculates \(\text{poissonpmf}(n, n)\) for values of \(n\) much larger than \(n^*\). Hint: For a Poisson (\(\alpha\)) random variable \(K\),

\[ P_K(k) = \exp \left( -\alpha + k \ln(\alpha) - \sum_{j=1}^{k} \ln(j) \right). \]
Quiz 4.8

Write a MATLAB function \( t = t2rv(m) \) that generates \( m \) samples of a random variable with the PDF \( f_{T|T>\theta}(t) \) as given in Example 7.10.

Problems

4.2.1 The cumulative distribution function of random variable \( X \) is
\[
F_X(x) = \begin{cases} 
0 & x < -1, \\
\frac{(x + 1)/2}{1} & -1 \leq x < 1, \\
1 & x \geq 1.
\end{cases}
\]

(a) What is \( P[X > 1/2]? \)
(b) What is \( P[-1/2 < X \leq 3/4]? \)
(c) What is \( P[|X| \leq 1/2]? \)
(d) What is the value of \( a \) such that \( P[X \leq a] = 0.8? \)

4.2.2 The CDF of the continuous random variable \( V \) is
\[
F_V(v) = \begin{cases} 
0 & v < -5, \\
c(v + 5)^2 & -5 \leq v < 7, \\
1 & v \geq 7.
\end{cases}
\]

(a) What is \( c? \)
(b) What is \( P[V > 4]? \)
(c) What is \( P[-3 < V \leq 0]? \)
(d) What is the value of \( a \) such that \( P[V > a] = 2/3? \)

4.2.3 In this problem, we verify that \( \lim_{n \to \infty} [nx]/n = x. \)
(a) Verify that \( nx \leq [nx] \leq nx + 1. \)
(b) Use part (a) to show
\[
\lim_{n \to \infty} [nx]/n = x.
\]
(c) Use a similar argument to show that \( \lim_{n \to \infty} [nx]/n = x. \)

4.2.4 The CDF of random variable \( W \) is
\[
F_W(w) = \begin{cases} 
0 & w < -5, \\
\frac{w+5}{8} & -5 \leq w < -3, \\
\frac{1}{4} & -3 \leq w < 3, \\
\frac{1}{4} + \frac{3(w-3)}{8} & 3 \leq w < 5, \\
1 & w \geq 5.
\end{cases}
\]

(a) What is \( P[W \leq 4]? \)
(b) What is \( P[-2 < W \leq 2]? \)
(c) What is \( P[W > 0]? \)
(d) What is the value of \( a \) such that \( P[W \leq a] = 1/2? \)

4.3.1 The random variable \( X \) has probability density function
\[
f_X(x) = \begin{cases} 
cx & 0 \leq x \leq 2, \\
0 & \text{otherwise}.
\end{cases}
\]

Use the PDF to find
(a) the constant \( c, \)
(b) \( P[0 \leq X \leq 1], \)
(c) \( P[-1/2 \leq X \leq 1/2], \)
(d) the CDF \( F_X(x). \)

4.3.2 The cumulative distribution function of random variable \( X \) is
\[
F_X(x) = \begin{cases} 
0 & x < -1, \\
(x + 1)/2 & -1 \leq x < 1, \\
1 & x \geq 1.
\end{cases}
\]

Find the PDF \( f_X(x) \) of \( X. \)

4.3.3 Find the PDF \( f_U(u) \) of the random variable \( U \) in Problem 4.2.4.
4.3.4 For a constant parameter $a > 0$, a Rayleigh random variable $X$ has PDF
\[ f_X(x) = \begin{cases} a^2xe^{-a^2x^2/2} & x > 0, \\ 0 & \text{otherwise}. \end{cases} \]

What is the CDF of $X$?

4.3.5 Random variable $X$ has a PDF of the form $f_X(x) = \frac{1}{2}f_1(x) + \frac{1}{2}f_2(x)$, where
\[ f_1(x) = \begin{cases} c_1 & 0 \leq x \leq 2, \\ 0 & \text{otherwise}, \end{cases} \]
\[ f_2(x) = \begin{cases} c_2e^{-x} & x \geq 0, \\ 0 & \text{otherwise}. \end{cases} \]

What conditions must $c_1$ and $c_2$ satisfy so that $f_X(x)$ is a valid PDF?

4.3.6 For constants $a$ and $b$, random variable $X$ has PDF
\[ f_X(x) = \begin{cases} ax^2 + bx & 0 \leq x \leq 1, \\ 0 & \text{otherwise}. \end{cases} \]

What conditions on $a$ and $b$ are necessary and sufficient to guarantee that $f_X(x)$ is a valid PDF?

4.4.1 Random variable $X$ has PDF
\[ f_X(x) = \begin{cases} 1/4 & -1 \leq x \leq 3, \\ 0 & \text{otherwise}. \end{cases} \]

Define the random variable $Y$ by $Y = h(X) = X^2$.
(a) Find $E[X]$ and $\text{Var}[X]$.
(b) Find $h(E[X])$ and $E[h(X)]$.
(c) Find $E[Y]$ and $\text{Var}[Y]$.

4.4.2 Let $X$ be a continuous random variable with PDF
\[ f_X(x) = \begin{cases} 1/8 & 1 \leq x \leq 9, \\ 0 & \text{otherwise}. \end{cases} \]

Let $Y = h(X) = 1/\sqrt{X}$.
(a) Find $E[X]$ and $\text{Var}[X]$.
(b) Find $h(E[X])$ and $E[h(X)]$.
(c) Find $E[Y]$ and $\text{Var}[Y]$.

4.4.3 Random variable $X$ has CDF
\[ F_X(x) = \begin{cases} 0 & x < 0, \\ x/2 & 0 \leq x \leq 2, \\ 1 & x > 2. \end{cases} \]

(a) What is $E[X]$?
(b) What is $\text{Var}[X]$?

4.4.4 The probability density function of random variable $Y$ is
\[ f_Y(y) = \begin{cases} y/2 & 0 \leq y < 2, \\ 0 & \text{otherwise}. \end{cases} \]

What are $E[Y]$ and $\text{Var}[Y]$?

4.4.5 The cumulative distribution function of the random variable $Y$ is
\[ F_Y(y) = \begin{cases} 0 & y < -1, \\ (y + 1)/2 & -1 \leq y \leq 1, \\ 1 & y > 1. \end{cases} \]

What are $E[Y]$ and $\text{Var}[Y]$?

4.4.6 The cumulative distribution function of random variable $V$ is
\[ F_V(v) = \begin{cases} 0 & v < -5, \\ (v + 5)^2/144 & -5 \leq v < 7, \\ 1 & v \geq 7. \end{cases} \]

(a) What are $E[V]$ and $\text{Var}[V]$?
(b) What is $E[V^3]$?

4.4.7 The cumulative distribution function of random variable $U$ is
\[ F_U(u) = \begin{cases} 0 & u < -5, \\ u + 3/8 & -5 \leq u < -3, \\ 1/4 & -3 \leq u < 3, \\ 3u - 7/8 & 3 \leq u < 5, \\ 1 & u \geq 5. \end{cases} \]

(a) What are $E[U]$ and $\text{Var}[U]$?
(b) What is $E[U^2]$?
4.4.8 $X$ is a Pareto $(\alpha, \mu)$ random variable, as defined in Appendix A. What is the largest value of $n$ for which the $n$th moment $E[X^n]$ exists? For all feasible values of $n$, find $E[X^n]$.

4.5.1 $Y$ is a continuous uniform $(1, 5)$ random variable.
(a) What is $P[Y > E[Y]]$?
(b) What is $P[Y \leq \text{Var}[Y]]$?

4.5.2 The current $Y$ across a 1 kΩ resistor is a continuous uniform $(−10, 10)$ random variable. Find $P[|Y| < 3]$.

4.5.3 Radars detect flying objects by measuring the power reflected from them. The reflected power of an aircraft can be modeled as a random variable $Y$ with PDF

$$f_Y(y) = \begin{cases} \frac{1}{P_0} e^{-y/P_0} & y \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

where $P_0 > 0$ is some constant. The aircraft is correctly identified by the radar if the reflected power of the aircraft is larger than its average value. What is the probability $P[C]$ that an aircraft is correctly identified?

4.5.4 $Y$ is an exponential random variable with variance $\text{Var}[Y] = 25$.
(a) What is the PDF of $Y$?
(b) What is $E[Y^2]$?
(c) What is $P[Y > 5]$?

4.5.5 The time delay $Y$ (in milliseconds) that your computer needs to connect to an access point is an exponential random variable.
(a) Find $P[Y > E[Y]]$.
(b) Find $P[Y > 2E[Y]]$.

4.5.6 $X$ is an Erlang $(n, \lambda)$ random variable with parameter $\lambda = 1/3$ and expected value $E[X] = 15$.
(a) What is the value of the parameter $n$?
(b) What is the PDF of $X$?

4.5.7 $Y$ is an Erlang $(n = 2, \lambda = 2)$ random variable.
(a) What is $E[Y]$?
(b) What is $\text{Var}[Y]$?
(c) What is $P[0.5 \leq Y < 1.5]$?

4.5.8 $U$ is a zero mean continuous uniform random variable. What is $P[U^2 \leq \text{Var}[U]]$?

4.5.9 $U$ is a continuous uniform random variable such that $E[U] = 10$ and $P[U > 12] = 1/4$. What is $P[U < 9]$?

4.5.10 $X$ is a continuous uniform $(-5, 5)$ random variable.
(a) What is the PDF $f_X(x)$?
(b) What is the CDF $F_X(x)$?
(c) What is $E[X]$?
(d) What is $E[X^5]$?
(e) What is $E[e^X]$?

4.5.11 $X$ is a continuous uniform $(-a, a)$ random variable. Find $P[|X| \leq \text{Var}[X]]$.

4.5.12 $X$ is a uniform random variable with expected value $\mu_X = 7$ and variance $\text{Var}[X] = 3$. What is the PDF of $X$?

4.5.13 The probability density function of random variable $X$ is

$$f_X(x) = \begin{cases} (1/2)e^{-x/2} & x \geq 0, \\ 0 & \text{otherwise}. \end{cases}$$

(a) What is $P[1 \leq X \leq 2]$?
(b) What is $F_X(x)$, the cumulative distribution function of $X$?
(c) What is $E[X]$, the expected value of $X$?
(d) What is $\text{Var}[X]$, the variance of $X$?

4.5.14 Verify parts (b) and (c) of Theorem 4.6 by directly calculating the expected value and variance of a uniform random variable with parameters $a < b$.

4.5.15 Long-distance calling plan $A$ offers flat-rate service at 10 cents per minute. Calling plan $B$ charges 99 cents for every call under 20 minutes; for calls over 20 minutes, the charge is 90 cents for the first 20
minute. (Note that these plans measure your call duration exactly, without rounding to the next minute or even second.) If your long-distance calls have exponential distribution with expected value \( \tau \) minutes, which plan offers a lower expected cost per call?

4.5.16 c In this problem we verify that an Erlang \((n, \lambda)\) PDF integrates to 1. Let the integral of the \(n\)th order Erlang PDF be denoted by

\[
I_n = \int_0^\infty \frac{\lambda^n x^{n-1} e^{-\lambda x}}{(n-1)!} \, dx.
\]

First, show directly that the Erlang PDF with \(n = 1\) integrates to 1 by verifying that \(I_1 = 1\). Second, use integration by parts (Appendix B, Math Fact B.10) to show that \(I_n = I_{n-1}\).

4.5.17 c Calculate the \(k\)th moment \(E[X^k]\) of an Erlang \((n, \lambda)\) random variable \(X\). Use your result to verify Theorem 4.10. Hint: Remember that the Erlang \((n + k, \lambda)\) PDF integrates to 1.

4.5.18 c In this problem, we outline the proof of Theorem 4.11.

(a) Let \(X_n\) denote an Erlang \((n, \lambda)\) random variable. Use the definition of the Erlang PDF to show that for any \(x \geq 0\),

\[
F_{X_n}(x) = \int_0^x \frac{\lambda^n t^{n-1} e^{-\lambda t}}{(n-1)!} \, dt.
\]

(b) Apply integration by parts (see Appendix B, Math Fact B.10) to this integral to show that for \(x \geq 0\),

\[
F_{X_n}(x) = F_{X_{n-1}}(x) - \frac{(\lambda x)^{n-1} e^{-\lambda x}}{(n-1)!}.
\]

(c) Use the fact that \(F_{X_1}(x) = 1 - e^{-\lambda x}\) for \(x \geq 0\) to verify the claim of Theorem 4.11.

4.5.20 c This problem outlines the steps needed to show that a nonnegative continuous random variable \(X\) has expected value

\[
E[X] = \int_0^\infty [1 - F_X(x)] \, dx.
\]

(a) For any \(r \geq 0\), show that

\[
rP[X > r] \leq \int_r^\infty x f_X(x) \, dx.
\]

(b) Use part (a) to argue that if \(E[X] < \infty\), then

\[
\lim_{r \to \infty} rP[X > r] = 0.
\]

(c) Now use integration by parts (Appendix B, Math Fact B.10) to evaluate

\[
\int_0^\infty [1 - F_X(x)] \, dx.
\]

4.6.1 c The peak temperature \(T\), as measured in degrees Fahrenheit, on a July day in New Jersey is the Gaussian \((85, 10)\) random variable. What is \(P[T > 100]\), \(P[T < 60]\), and \(P[70 \leq T \leq 100]\)?

4.6.2 c What is the PDF of \(Z\), the standard normal random variable?

4.6.3 c Find each probability.

(a) \(V\) is a Gaussian \((\mu = 0, \sigma = 2)\) random variable. Find \(P[V > 4]\).

(b) \(W\) is a Gaussian \((\mu = 2, \sigma = 5)\) random variable. What is \(P[W \leq 2]\)?

(c) For a Gaussian \((\mu, \sigma = 2)\) random variable \(X\), find \(P[X \leq \mu + 1]\).

(d) \(Y\) is a Gaussian \((\mu = 50, \sigma = 10)\) ran-
4.6.4 In each of the following cases, \( Y \) is a Gaussian random variable. Find the expected value \( \mu = E[Y] \).

(a) \( Y \) has standard deviation \( \sigma = 10 \) and \( P[Y \leq 10] = 0.933 \).

(b) \( Y \) has standard deviation \( \sigma = 10 \) and \( P[Y \leq 0] = 0.067 \).

(c) \( Y \) has standard deviation \( \sigma \) and \( P[Y \leq 10] = 0.977 \). (Find \( \mu \) as a function of \( \sigma \).)

(d) \( P[Y > 5] = 1/2 \).

4.6.5 Your internal body temperature \( T \) in degrees Fahrenheit is a Gaussian \((\mu = 98.6, \sigma = 0.4)\) random variable. In terms of the \( \Phi(\cdot) \) function, find \( P[T > 100] \). Does this model seem reasonable?

4.6.6 The temperature \( T \) in this thermostatically controlled lecture hall is a Gaussian random variable with expected value \( \mu = 68 \) degrees Fahrenheit. In addition, \( P[T < 66] = 0.1587 \). What is the variance of \( T \)?

4.6.7 \( X \) is a Gaussian random variable with \( E[X] = 0 \) and \( P[|X| \leq 10] = 0.1 \). What is the standard deviation \( \sigma_X \)?

4.6.8 A function commonly used in communications textbooks for the tail probabilities of Gaussian random variables is the complementary error function, defined as

\[
erfc(z) = \frac{2}{\sqrt{\pi}} \int_z^\infty e^{-x^2} \, dx.
\]

Show that

\[
Q(z) = \frac{1}{2} \erfc\left(\frac{z}{\sqrt{2}}\right).
\]

4.6.9 The peak temperature \( T \), in degrees Fahrenheit, on a July day in Antarctica is a Gaussian random variable with a variance of 225. With probability 1/2, the temperature \( T \) exceeds -75 degrees. What is \( P[T > 0] \)? What is \( P[T < -100] \)?

4.6.10 A professor pays 25 cents for each blackboard error made in lecture to the student who points out the error. In a career of \( n \) years filled with blackboard errors, the total amount in dollars paid can be approximated by a Gaussian random variable \( Y_n \) with expected value \( 40n \) and variance 100\( n \). What is the probability that \( Y_{20} \) exceeds 1000? How many years \( n \) must the professor teach in order that \( P[Y_n > 1000] > 0.99 \)?

4.6.11 Suppose that out of 100 million men in the United States, 23,000 are at least 7 feet tall. Suppose that the heights of U.S. men are independent Gaussian random variables with a expected value of 5'10". Let \( N \) equal the number of men who are at least 7'6" tall.

(a) Calculate \( \sigma_X \), the standard deviation of the height of U.S. men.

(b) In terms of the \( \Phi(\cdot) \) function, what is the probability that a randomly chosen man is at least 8 feet tall?

(c) What is the probability that no man alive in the United States today is at least 7'6" tall?

(d) What is \( E[N] \)?

4.6.12 In this problem, we verify that for \( x \geq 0 \),

\[
\Phi(x) = \frac{1}{2} + \frac{1}{2} \erf\left(\frac{x}{\sqrt{2}}\right).
\]

(a) Let \( Y \) have a Gaussian \((0, 1/\sqrt{2})\) distribution and show that

\[
F_Y(y) = \int_{-\infty}^y f_Y(u) \, du = \frac{1}{2} + \erf(y).
\]

(b) Observe that \( Z = \sqrt{2}Y \) is Gaussian \((0, 1)\) and show that

\[
\Phi(z) = F_Z(z) = F_Y\left(\frac{z}{\sqrt{2}}\right).
\]

4.6.13 This problem outlines the steps needed to show that the Gaussian PDF integrates to unity. For a Gaussian \((\mu, \sigma)\) random variable \( W \), we will show that

\[
I = \int_{-\infty}^{\infty} f_W(w) \, dw = 1.
\]
(a) Use the substitution \( x = (w - \mu)/\sigma \) to show that
\[
I = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/2} \, dx.
\]

(b) Show that
\[
I^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(x^2+y^2)/2} \, dx \, dy.
\]

(c) Change to polar coordinates to show that \( I^2 = 1 \).

**4.6.14** At time \( t = 0 \), the price of a stock is a constant \( k \) dollars. At time \( t > 0 \) the price of a stock is a Gaussian random variable \( X \) with \( \text{E}[X] = k \) and \( \text{Var}[X] = t \). At time \( t \), a Call Option at Strike \( k \) has value \( V = (X - k)^+ \), where the operator \((\cdot)^+\) is defined as \((z)^+ = \max(z, 0)\).

(a) Find the expected value \( \text{E}[V] \).

(b) Suppose you can buy the call option for \( d \) dollars at time \( t = 0 \). At time \( t \), you can sell the call for \( V \) dollars and earn a profit (or loss perhaps) of \( R = V - d \) dollars. Let \( d_0 \) denote the value of \( d \) such that \( \text{P}[R > 0] = 1/2 \). Your strategy is to buy the option if \( d \leq d_0 \) so that your probability of a profit is \( \text{P}[R > 0] \geq 1/2 \). Find \( d_0 \).

(c) Let \( d_1 \) denote the value of \( d \) such that \( \text{E}[R] = 0.01 \times d \). Now your strategy is to buy the option if \( d \leq d_1 \) so that your expected return is at least one percent of the option cost. Find \( d_1 \).

(d) Are the strategies “Buy the option if \( d \leq d_0 \)” and “Buy the option if \( d \leq d_1 \)” reasonable strategies?

**4.6.15** In mobile radio communications, the radio channel can vary randomly. In particular, in communicating with a fixed transmitter power over a “Rayleigh fading” channel, the receiver signal-to-noise ratio \( Y \) is an exponential random variable with expected value \( \gamma \). Moreover, when \( Y = y \), the probability of an error in decoding a transmitted bit is \( P_e(y) = Q(\sqrt{2\gamma}) \) where \( Q(\cdot) \) is the standard normal complementary CDF.

The average probability of bit error, also known as the bit error rate or BER, is
\[
\bar{P}_e = \text{E}[P_e(Y)] = \int_{-\infty}^{\infty} Q(\sqrt{2\gamma}) f_Y(y) \, dy.
\]

Find a simple formula for the BER \( \bar{P}_e \) as a function of the average SNR \( \gamma \).

**4.6.16** At time \( t = 0 \), the price of a stock is a constant \( k \) dollars. At some future time \( t > 0 \), the price \( X \) of the stock is a uniform \((k-t, k+t)\) random variable. At this time \( t \), a Put Option at Strike \( k \) (the right to sell the stock at price \( k \)) has value \((k-X)^+ \) dollars where the operator \((\cdot)^+\) is defined as \((z)^+ = \max(z, 0)\). Similarly a Call Option at Strike \( k \) (the right to buy the stock at price \( k \)) at time \( t \) has value \((X-k)^+ \).

(a) At time 0, you sell the put and receive \( d \) dollars. At time \( t \), you purchase the put for \((k-X)^+ \) dollars to cancel your position. Your gain is
\[
R = g_p(X) = d - (k - X)^+.
\]

Find the central moments \( \text{E}[R] \) and \( \text{Var}[R] \).

(b) In a short straddle, you sell the put for \( d \) dollars and you also sell the call for \( d \) dollars. At a future time \( t > 0 \), you purchase the put for \((k-X)^+ \) dollars and the call for \((X-k)^+ \) dollars to cancel both positions. Your gain on the put is \( g_p(X) = d - (k - X)^+ \) dollars and your gain on the call is \( g_c(X) = d - (X - k)^+ \) dollars. Your net gain is
\[
R' = g_p(X) + g_c(X).
\]

Find the expected value \( \text{E}[R'] \) and variance \( \text{Var}[R'] \).

(c) Explain why selling the straddle might be attractive compared to selling just the put or just the call.

**4.6.17** Continuing Problem 4.6.16, suppose you sell the straddle at time \( t = 0 \) and liquidate your position at time \( t \), generating a profit (or perhaps a loss) \( R' \). Find the
PDF $f_{R'}(r)$ of $R'$. Suppose $d$ is sufficiently large that $E[R'] > 0$. Would you be interested in selling the short straddle? Are you getting something, namely $E[R']$ dollars, for nothing?

4.7.1 Let $X$ be a random variable with CDF

$$F_X(x) = \begin{cases} 
0 & x < -1, \\
\frac{x}{3} + \frac{1}{3} & -1 \leq x < 0, \\
\frac{x}{3} + \frac{2}{3} & 0 \leq x < 1, \\
1 & 1 \leq x.
\end{cases}$$

Sketch the CDF and find

(a) $P[X < -1]$ and $P[X \leq -1]$,
(b) $P[X < 0]$ and $P[X \leq 0]$,
(c) $P[0 < X \leq 1]$ and $P[0 \leq X \leq 1]$.

4.7.2 Let $X$ be a random variable with CDF

$$F_X(x) = \begin{cases} 
0 & x < -1, \\
\frac{x}{4} + \frac{1}{2} & -1 \leq x < 1, \\
1 & 1 \leq x.
\end{cases}$$

Sketch the CDF and find

(a) $P[X < -1]$ and $P[X \leq -1]$,
(b) $P[X < 0]$ and $P[X \leq 0]$,
(c) $P[X > 1]$ and $P[X \geq 1]$.

4.7.3 For random variable $X$ of Problem 4.7.2, find $f_X(x)$, $E[X]$, and $\text{Var}[X]$.

4.7.4 $X$ is a Bernoulli random variable with expected value $p$. What is the PDF $f_X(x)$?

4.7.5 $X$ is a geometric random variable with expected value $1/p$. What is the PDF $f_X(x)$?

4.7.6 When you make a phone call, the line is busy with probability 0.2 and no one answers with probability 0.3. The random variable $X$ describes the conversation time (in minutes) of a phone call that is answered. $X$ is an exponential random variable with $E[X] = 3$ minutes. Let the random variable $W$ denote the conversation time (in seconds) of all calls ($W = 0$ when

(a) What is $f_W(w)$?
(b) What is $f_W(w)$?
(c) What are $E[W]$ and $\text{Var}[W]$?

4.7.7 For 80% of lectures, Professor X arrives on time and starts lecturing with delay $T = 0$. When Professor X is late, the starting time delay $T$ is uniformly distributed between 0 and 300 seconds. Find the CDF and PDF of $T$.

4.7.8 With probability 0.7, the toss of an Olympic shot-putter travels $D = 60 + X$ feet, where $X$ is an exponential random variable with expected value $\mu = 10$. Otherwise, with probability 0.3, a foul is committed by stepping outside of the shot-put circle and we say $D = 0$. What are the CDF and PDF of random variable $D$?

4.7.9 For 70% of lectures, Professor Y arrives on time. When Professor Y is late, the arrival time delay is a continuous random variable uniformly distributed from 0 to 10 minutes. Yet, as soon as Professor Y is 5 minutes late, all the students get up and leave. (It is unknown if Professor Y still conducts the lecture.) If a lecture starts when Professor Y arrives and always ends 80 minutes after the scheduled starting time, what is the PDF of $T$, the length of time that the students observe a lecture.

4.8.1 Write a function $y=\text{quiz31rv}(m)$ that produces $m$ samples of random variable $Y$ defined in Quiz 4.2.

4.8.2 For the Gaussian $(0, 1)$ complementary CDF $Q(z)$, a useful numerical approximation for $z \geq 0$ is

$$Q(z) \approx \left( \sum_{n=1}^{5} a_n t^n \right) e^{-z^2/2},$$

where

$$t = \frac{1}{1 + 0.231641888 z}, \quad a_1 = 0.12741
d_{96} \quad a_2 = -0.142248368 \quad a_3 = 0.7107068705$$
To compare this approximation to $Q(z)$, use MATLAB to graph

$$e(z) = \frac{Q(z) - \hat{Q}(z)}{Q(z)}.$$ 

4.8.3 Use exponentialrv.m and Theorem 4.9 and to write a MATLAB function `k=georv(p,m)` that generates $m$ samples of a geometric ($p$) random variable $K$. Compare the resulting algorithm to the technique employed in Problem 3.9.8 for `geometricrv(p,m).

4.8.4 Applying Equation (4.14) with $x$ replaced by $i\Delta$ and $dx$ replaced by $\Delta$, we obtain

$$P[i\Delta < X \leq i\Delta + \Delta] = f_X(i\Delta) \Delta.$$ 

If we generate a large number $n$ of samples of random variable $X$, let $n_i$ denote the number of occurrences of the event

$$\{i\Delta < X \leq (i+1)\Delta\}.$$ 

We would expect that

$$\lim_{n \to \infty} \frac{n_i}{n} = f_X(i\Delta) \Delta,$$

or equivalently,

$$\lim_{n \to \infty} \frac{n_i}{n\Delta} = f_X(i\Delta).$$

Use MATLAB to confirm this with $\Delta = 0.01$ for

(a) an exponential ($\lambda = 1$) random variable $X$ and for $i = 0, \ldots, 500,

(b) a Gaussian (3,1) random variable $X$ and for $i = 0, \ldots, 600.$
The sample output of `gauss2var` shown here is produced with the commands

```matlab
>> xy=gauss2rv(3,3,5,1,0.5,500);
>> plot(xy(:,1),xy(:,2),'.');
```

We observe that the center of the cloud is \((\mu_X, \mu_Y) = (3,5)\). In addition, we note that the \(X\) and \(Y\) axes are scaled differently because \(\sigma_X = 3\) and \(\sigma_Y = 1\).

We observe that this example with \(\rho_{X,Y} = 0.5\) shows random variables that are less correlated than the examples in Figure 5.5 with \(|\rho| = 0.9\).

We note that bivariate Gaussian random variables are a special case of \(n\)-dimensional Gaussian random vectors, which are introduced in Chapter 8. Based on linear algebra techniques, Chapter 8 introduces the `gaussvector` function to generate samples of Gaussian random vectors that generalizes `gauss2rv` to \(n\) dimensions.

Beyond bivariate Gaussian pairs, there exist a variety of techniques for generating sample values of pairs of continuous random variables of specific types. A basic approach is to generate \(X\) based on the marginal PDF \(f_X(x)\) and then generate \(Y\) using a conditional probability model that depends on the value of \(X\). Conditional probability models and MATLAB techniques that employ these models are the subject of Chapter 7.

**Problems**

<table>
<thead>
<tr>
<th>Difficulty:</th>
<th>Easy</th>
<th>Moderate</th>
<th>Difficult</th>
<th>Experts Only</th>
</tr>
</thead>
</table>

**5.1.1** Random variables \(X\) and \(Y\) have the joint CDF

\[
F_{X,Y}(x,y) = \begin{cases} 
(1 - e^{-x})(1 - e^{-y}) & x \geq 0; \\
0 & \text{ow}.
\end{cases}
\]

(a) What is \(P[X \leq 2, Y \leq 3]\)?
(b) What is the marginal CDF, \(F_X(x)\)?
(c) What is the marginal CDF, \(F_Y(y)\)?

**5.1.2** Express the following extreme values of \(F_{X,Y}(x,y)\) in terms of the marginal cumulative distribution functions \(F_X(x)\) and \(F_Y(y)\).

(a) \(F_{X,Y}(x, -\infty)\)
(b) \(F_{X,Y}(x, \infty)\)
(c) \(F_{X,Y}(-\infty, \infty)\)
(d) \(F_{X,Y}(-\infty, y)\)

**5.1.3** For continuous random variables \(X, Y\) with joint CDF \(F_{X,Y}(x,y)\) and marginal CDFs \(F_X(x)\) and \(F_Y(y)\), find \(P[x_1 \leq X < x_2 \cup y_1 \leq Y < y_2]\). This is the probability of the shaded "cross" region in the following diagram.
5.1.4 Random variables \( X \) and \( Y \) have CDF \( F_X(x) \) and \( F_Y(y) \). Is \( F(x, y) = F_X(x)F_Y(y) \) a valid CDF? Explain your answer.

5.1.5 In this problem, we prove Theorem 5.2.
(a) Sketch the following events on the \( X, Y \) plane:
\[
A = \{ X \leq x_1, y_1 < Y \leq y_2 \}, \\
B = \{ x_1 < X \leq x_2, Y \leq y_1 \}, \\
C = \{ x_1 < X \leq x_2, y_1 < Y \leq y_2 \}.
\]
(b) Express the probability of the events \( A, B, \) and \( A \cup B \cup C \) in terms of the joint CDF \( F_{X,Y}(x, y) \).
(c) Use the observation that events \( A, B, \) and \( C \) are mutually exclusive to prove Theorem 5.2.

5.1.6 Can the following function be the joint CDF of random variables \( X \) and \( Y \)?
\[
F(x, y) = \begin{cases} 
1 - e^{-(x+y)} & x \geq 0, y \geq 0, \\
0 & \text{otherwise}.
\end{cases}
\]

5.2.1 Random variables \( X \) and \( Y \) have the joint PMF
\[
P_{X,Y}(x, y) = \begin{cases} 
\frac{cxy}{x=1,2,4; y=1,3,} \\
0 & \text{otherwise}.
\end{cases}
\]
(a) What is the value of the constant \( c \)?
(b) What is \( P[Y < X] \)?
(c) What is \( P[Y > X] \)?
(d) What is \( P[Y = X] \)?
(e) What is \( P[Y = 3] \)?

5.2.2 Random variables \( X \) and \( Y \) have the joint PMF
\[
P_{X,Y}(x, y) = \begin{cases} 
c|x+y| & x = -2, 0, 2; y = -1, 0, 1, \\
0 & \text{otherwise}.
\end{cases}
\]
(a) What is the value of the constant \( c \)?
(b) What is \( P[Y < X] \)?
(c) What is \( P[Y > X] \)?
(d) What is \( P[Y = X] \)?
(e) What is \( P[X < 1] \)?

5.2.3 Test two integrated circuits. In each test, the probability of rejecting the circuit is \( p \), independent of the other test. Let \( X \) be the number of rejects (either 0 or 1) in the first test and let \( Y \) be the number of rejects in the second test. Find the joint PMF \( P_{X,Y}(x, y) \).

5.2.4 For two independent flips of a fair coin, let \( X \) equal the total number of tails and let \( Y \) equal the number of heads on the last flip. Find the joint PMF \( P_{X,Y}(x, y) \).

5.2.5 In Figure 5.2, the axes of the figures are labeled \( X \) and \( Y \) because the figures depict possible values of the random variables \( X \) and \( Y \). However, the figure at the end of Example 5.3 depicts \( P_{X,Y}(x, y) \) on axes labeled with lowercase \( x \) and \( y \). Should those axes be labeled with the uppercase \( X \) and \( Y \)? Hint: Reasonable arguments can be made for both views.

5.2.6 As a generalization of Example 5.3, consider a test of \( n \) circuits such that each circuit is acceptable with probability \( p \), independent of the outcome of any other test. Show that the joint PMF of \( X \), the number of acceptable circuits, and \( Y \), the number of acceptable circuits found before observing the first reject, is
\[
P_{X,Y}(x, y) = \begin{cases} 
\binom{n-1}{x} p^x (1-p)^{n-x} & 0 \leq y \leq x < n, \\
p^n & x = y = n, \\
0 & \text{otherwise}.
\end{cases}
\]
Hint: For \( 0 \leq y \leq x < n \), show that
\[
\{X = x, Y = y\} = A \cap B \cap C,
\]
where
- \( A \): The first \( y \) tests are acceptable.
- \( B \): Test \( y + 1 \) is a rejection.
- \( C \): The remaining \( n - y - 1 \) tests yield \( x - y \) acceptable circuits.
5.2.7 With two minutes left in a five-minute overtime, the score is 0–0 in a Rutgers soccer match versus Villanova. (Note that the overtime is NOT sudden-death.) In the next-to-last minute of the game, either (1) Rutgers scores a goal with probability \( p = 0.2 \), (2) Villanova scores with probability \( p = 0.2 \), or (3) neither team scores with probability \( 1 - 2p = 0.6 \). If neither team scores in the next-to-last minute, then in the final minute, either (1) Rutgers scores a goal with probability \( q = 0.3 \), (2) Villanova scores with probability \( q = 0.3 \), or (3) neither team scores with probability \( 1 - 2q = 0.4 \). However, if a team scores in the next-to-last minute, the trailing team goes for broke so that in the last minute, either (1) the leading team scores with probability 0.5, or (2) the trailing team scores with probability 0.5. For the final two minutes of overtime:

(a) Sketch a probability tree and construct a table for \( P_{R,V}(r,v) \), the joint PMF of \( R \), the number of Rutgers goals scored, and \( V \), the number of Villanova goals scored.

(b) What is the probability \( P[T] \) that the overtime ends in a tie?

(c) What is the PMF of \( R \), the number of goals scored by Rutgers?

(d) What is the PMF of \( G \), the total number of goals scored?

5.2.8 Each test of an integrated circuit produces an acceptable circuit with probability \( p \), independent of the outcome of the test of any other circuit. In testing \( n \) circuits, let \( K \) denote the number of circuits rejected and let \( X \) denote the number of acceptable circuits (either 0 or 1) in the last test. Find the joint PMF \( P_{K,X}(k,x) \).

5.2.9 Each test of an integrated circuit produces an acceptable circuit with probability \( p \), independent of the outcome of the test of any other circuit. In testing \( n \) circuits, let \( K \) denote the number of circuits rejected and let \( X \) denote the number of acceptable circuits that appear before the first reject is found. Find the joint PMF \( P_{K,X}(k,x) \).

5.3.1 Given the random variables \( X \) and \( Y \) in Problem 5.2.1, find

(a) The marginal PMFs \( P_X(x) \) and \( P_Y(y) \).

(b) The expected values \( E[X] \) and \( E[Y] \).

(c) The standard deviations \( \sigma_X \) and \( \sigma_Y \).

5.3.2 Given the random variables \( X \) and \( Y \) in Problem 5.2.2, find

(a) The marginal PMFs \( P_X(x) \) and \( P_Y(y) \).

(b) The expected values \( E[X] \) and \( E[Y] \).

(c) The standard deviations \( \sigma_X \) and \( \sigma_Y \).

5.3.3 For \( n = 0, 1, \ldots \) and \( 0 \leq k \leq 100 \), the joint PMF of random variables \( N \) and \( K \) is

\[
P_{N,K}(n,k) = \frac{100^ne^{-100}}{n!} \binom{100}{k} p^k (1-p)^{100-k}.
\]

Otherwise, \( P_{N,K}(n,k) = 0 \). Find the marginal PMFs \( P_N(n) \) and \( P_K(k) \).

5.3.4 Random variables \( X \) and \( Y \) have joint PMF

\[
P_{X,Y}(x,y) = \begin{cases} 
1/21 & x = 0, 1, 2, 3, 4, 5; \quad y = 0, 1, \ldots, x, \\
0 & \text{otherwise.}
\end{cases}
\]

Find the marginal PMFs \( P_X(x) \) and \( P_Y(y) \) and the expected values \( E[X] \) and \( E[Y] \).

5.3.5 Random variables \( N \) and \( K \) have the joint PMF

\[
P_{N,K}(n,k) = \begin{cases} 
(1-p)^{k-1} p \frac{n!}{k!} & k=1, \ldots, n; \quad n=1,2,\ldots, \\
0 & \text{otherwise.}
\end{cases}
\]

Find the marginal PMFs \( P_N(n) \) and \( P_K(k) \).

5.3.6 Random variables \( N \) and \( K \) have the joint PMF

\[
P_{N,K}(n,k) = \begin{cases} 
\frac{100^ne^{-100}}{(n+1)!} & k=0,1,\ldots,n; \quad n=0,1,\ldots, \\
0 & \text{otherwise.}
\end{cases}
\]
Find the marginal PMF $P_N(n)$. Show that the marginal PMF $P_k(k)$ satisfies $P_k(k) = P[N > k]/100$.

### 5.4.1 Random variables $X$ and $Y$ have joint PDF

$$f_{X,Y}(x,y) = \begin{cases} c & x \geq 0, y \geq 0, x + y \leq 1, \\ 0 & \text{otherwise.} \end{cases}$$

(a) What is the value of the constant $c$?
(b) What is $P[X \leq Y]$?
(c) What is $P[X + Y \leq 1/2]$?

### 5.4.2 Random variables $X$ and $Y$ have joint PDF

$$f_{X,Y}(x,y) = \begin{cases} cxy^2 & 0 \leq x \leq 1, 0 \leq y \leq 1, \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find the constant $c$.
(b) Find $P[X > Y]$ and $P[Y < X^2]$.
(c) Find $P[\min(X,Y) \leq 1/2]$.
(d) Find $P[\max(X,Y) \leq 3/4]$.

### 5.4.3 Random variables $X$ and $Y$ have joint PDF

$$f_{X,Y}(x,y) = \begin{cases} 6e^{-2x+3y} & x \geq 0, y \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find $P[X > Y]$ and $P[X + Y \leq 1]$.
(b) Find $P[\min(X,Y) \geq 1]$.
(c) Find $P[\max(X,Y) \leq 1]$.

### 5.4.4 Random variables $X$ and $Y$ have joint PDF

$$f_{X,Y}(x,y) = \begin{cases} 8xy & 0 \leq y \leq x \leq 1, \\ 0 & \text{otherwise.} \end{cases}$$

Following the method of Example 5.8, find the joint CDF $F_{X,Y}(x,y)$.

### 5.5.1 Random variables $X$ and $Y$ have joint PDF

$$f_{X,Y}(x,y) = \begin{cases} 1/2 & -1 \leq x \leq y \leq 1, \\ 0 & \text{otherwise.} \end{cases}$$

(a) What is the marginal PDF $f_X(x)$?
(b) What is the marginal PDF $f_Y(y)$?

Sketch the region of nonzero probability and answer the following questions.

(a) What is $P[X > 0]$?
(b) What is $f_X(x)$?
(c) What is $E[X]$?

### 5.5.2 Random variables $X$ and $Y$ have joint PDF

$$f_{X,Y}(x,y) = \begin{cases} cx & 0 \leq x \leq 1, 0 \leq y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

(a) Find the constant $c$.
(b) Find the marginal PDF $f_X(x)$.
(c) Are $X$ and $Y$ independent? Justify your answer.

### 5.5.3 $X$ and $Y$ are random variables with the joint PDF

$$f_{X,Y}(x,y) = \begin{cases} 2 & x + y \leq 1, x \geq 0, y \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

(a) What is the marginal PDF $f_X(x)$?
(b) What is the marginal PDF $f_Y(y)$?

### 5.5.4 Over the circle $X^2 + Y^2 \leq r^2$, random variables $X$ and $Y$ have the uniform PDF

$$f_{X,Y}(x,y) = \begin{cases} 1/(\pi r^2) & x^2 + y^2 \leq r^2, \\ 0 & \text{otherwise.} \end{cases}$$

(a) What is the marginal PDF $f_X(x)$?
(b) What is the marginal PDF $f_Y(y)$?

### 5.5.5 $X$ and $Y$ are random variables with the joint PDF

$$f_{X,Y}(x,y) = \begin{cases} 5x^2/2 & -1 \leq x \leq 1; \\ 0 & 0 \leq y \leq x^2 \text{ otherwise.} \end{cases}$$

(a) What is the marginal PDF $f_X(x)$?
(b) What is the marginal PDF $f_Y(y)$?
5.5.6 Over the circle \( X^2 + Y^2 \leq r^2 \), random variables \( X \) and \( Y \) have the PDF
\[
f_{X,Y}(x, y) = \begin{cases} \frac{2|xy|}{r^4} & x^2 + y^2 \leq r^2, \\ 0 & \text{otherwise}. \end{cases}
\]

(a) What is the marginal PDF \( f_X(x) \)?
(b) What is the marginal PDF \( f_Y(y) \)?

5.5.7 For a random variable \( X \), let \( Y = aX + b \). Show that if \( a > 0 \) then \( \rho_{X,Y} = 1 \).
Also show that if \( a < 0 \), then \( \rho_{X,Y} = -1 \).

5.5.8 Random variables \( X \) and \( Y \) have joint PDF
\[
f_{X,Y}(x, y) = \begin{cases} \frac{(x+y)}{3} & 0 \leq x \leq 1, \\ 0 & 0 \leq y \leq 2, \\ 0 & \text{otherwise}. \end{cases}
\]

(a) Find the marginal PDFs \( f_X(x) \) and \( f_Y(y) \).
(b) What are \( E[X] \) and \( \text{Var}[X] \)?
(c) What are \( E[Y] \) and \( \text{Var}[Y] \)?

5.5.9 Random variables \( X \) and \( Y \) have the joint PDF
\[
f_{X,Y}(x, y) = \begin{cases} cy & 0 \leq y \leq x \leq 1, \\ 0 & \text{otherwise}. \end{cases}
\]

(a) Draw the region of nonzero probability.
(b) What is the value of the constant \( c \)?
(c) What is \( F_X(x) \)?
(d) What is \( F_Y(y) \)?
(e) What is \( P[Y \leq X/2] \)?

5.6.1 An ice cream company needs to order ingredients from its suppliers. Depending on the size of the order, the weight of the shipment can be either
- 1 kg for a small order,
- 2 kg for a big order.

The company has three different suppliers. The vanilla supplier is 20 miles away. The chocolate supplier is 100 miles away. The strawberry supplier is 300 miles away. An experiment consists of monitoring an order and observing \( W \), the weight of the order, and \( D \), the distance the shipment must be sent. The following probability model describes the experiment:

<table>
<thead>
<tr>
<th></th>
<th>van.</th>
<th>choc.</th>
<th>straw.</th>
</tr>
</thead>
<tbody>
<tr>
<td>small</td>
<td>0.2</td>
<td>0.2</td>
<td>0.2</td>
</tr>
<tr>
<td>big</td>
<td>0.1</td>
<td>0.2</td>
<td>0.1</td>
</tr>
</tbody>
</table>

(a) What is the joint PMF \( P_{W,D}(w, d) \) of the weight and the distance?
(b) Find the expected shipping distance \( E[D] \).
(c) Are \( W \) and \( D \) independent?

5.6.2 A company receives shipments from two factories. Depending on the size of the order, a shipment can be in
- 1 box for a small order,
- 2 boxes for a medium order,
- 3 boxes for a large order.

The company has two different suppliers. Factory Q is 60 miles from the company. Factory R is 180 miles from the company. An experiment consists of monitoring a shipment and observing \( B \), the number of boxes, and \( M \), the number of miles the shipment travels. The following probability model describes the experiment:

<table>
<thead>
<tr>
<th></th>
<th>Factory Q</th>
<th>Factory R</th>
</tr>
</thead>
<tbody>
<tr>
<td>small order</td>
<td>0.3</td>
<td>0.2</td>
</tr>
<tr>
<td>medium order</td>
<td>0.1</td>
<td>0.2</td>
</tr>
<tr>
<td>large order</td>
<td>0.1</td>
<td>0.1</td>
</tr>
</tbody>
</table>

(a) Find \( P_{B,M}(b, m) \), the joint PMF of the number of boxes and the distance.
(b) What is \( E[B] \), the expected number of boxes?
(c) Are \( B \) and \( M \) independent?

5.6.3 Observe 100 independent flips of a fair coin. Let \( X \) equal the number of heads in the first 75 flips. Let \( Y \) equal the number of heads in the remaining 25 flips. Find \( P_X(x) \) and \( P_Y(y) \). Are \( X \) and \( Y \) independent? Find \( P_{X,Y}(x, y) \).

5.6.4 Observe independent flips of a fair coin until heads occurs twice. Let \( X_1 \) equal the number of flips up to and including the
first $H$. Let $X_2$ equal the number of additional flips up to and including the second $H$. What are $P_{X_1}(x_1)$ and $P_{X_2}(x_2)$? Are $X_1$ and $X_2$ independent? Find $P_{X_1, X_2}(x_1, x_2)$.

**5.6.5** $\triangleright$ $X$ is the continuous uniform $(0, 2)$ random variable. $Y$ has the continuous uniform $(0, 5)$ PDF, independent of $X$. What is the joint PDF $f_{X,Y}(x, y)$?

**5.6.6** $\triangleright$ $X_1$ and $X_2$ are independent random variables such that $X_1$ has PDF

$$f_{X_1}(x) = \begin{cases} \lambda_1 e^{-\lambda_1 x} & x \geq 0, \\ 0 & \text{otherwise}. \end{cases}$$

What is $P[X_2 < X_1]$?

**5.6.7** $\triangleright$ In terms of a positive constant $k$, random variables $X$ and $Y$ have joint PDF

$$f_{X,Y}(x, y) = \begin{cases} k + 3x^2 & -1/2 \leq x \leq 1/2, \\ -1/2 \leq y \leq 1/2, \\ 0 & \text{otherwise}. \end{cases}$$

(a) What is $k$?

(b) What is the marginal PDF of $X$?

(c) What is the marginal PDF of $Y$?

(d) Are $X$ and $Y$ independent?

**5.6.8** $\triangleright$ $X_1$ and $X_2$ are independent, identically distributed random variables with PDF

$$f_X(x) = \begin{cases} x/2 & 0 \leq x \leq 2, \\ 0 & \text{otherwise}. \end{cases}$$

(a) Find the CDF, $F_X(x)$.

(b) What is $P[X_1 \leq 1, X_2 \leq 1]$, the probability that $X_1$ and $X_2$ are both less than or equal to 1?

(c) Let $W = \max(X_1, X_2)$. What is $F_W(1)$, the CDF of $W$ evaluated at $w = 1$?

(d) Find the CDF $F_W(w)$.

**5.6.9** $\triangleright$ Prove that random variables $X$ and $Y$ are independent if and only if

$$f_{X,Y}(x, y) = f_X(x)f_Y(y).$$

**5.7.1** $\triangleright$ Continuing Problem 5.6.1, the price per kilogram for shipping the order is one cent per mile. $C$ cents is the shipping cost of one order. What is $E[C]$?

**5.7.2** $\triangleright$ Continuing Problem 5.6.2, the price per mile of shipping each box is one cent per mile the box travels. $C$ cents is the price of one shipment. What is $E[C]$, the expected price of one shipment?

**5.7.3** $\triangleright$ A random ECE sophomore has height $X$ (rounded to the nearest foot) and GPA $Y$ (rounded to the nearest integer). These random variables have joint PMF

$$P_{X,Y}(x, y) = \begin{array}{cccc} x & y = 1 & y = 2 & y = 3 & y = 4 \\ x = 5 & 0.05 & 0.1 & 0.2 & 0.05 \\ x = 6 & 0.1 & 0.3 & 0.1 & \end{array}$$

Find $E[X + Y]$ and $\text{Var}[X + Y]$.

**5.7.4** $\triangleright$ $X$ and $Y$ are independent, identically distributed random variables with PMF

$$P_X(k) = P_Y(k) = \begin{cases} 3/4 & k = 0, \\ 1/4 & k = 20, \\ 0 & \text{otherwise}. \end{cases}$$

Find the following quantities:

$$E[X], \quad \text{Var}[X], \quad E[X + Y], \quad \text{Var}[X + Y], \quad \text{E}[XY^{2XY}].$$

**5.7.5** $\triangleright$ $X$ and $Y$ are random variables with $E[X] = E[Y] = 0$ and $\text{Var}[X] = 1$, $\text{Var}[Y] = 4$ and correlation coefficient $\rho = 1/2$. Find $\text{Var}[X + Y]$.

**5.7.6** $\triangleright$ $X$ and $Y$ are random variables such that $X$ has expected value $\mu_X = 0$ and standard deviation $\sigma_X = 3$ while $Y$ has expected value $\mu_Y = 1$ and standard deviation $\sigma_Y = 4$. In addition, $X$ and $Y$ have covariance $\text{Cov}[X, Y] = -3$. Find the expected value and variance of $W = 2X + 2Y$.

**5.7.7** $\triangleright$ Observe independent flips of a fair coin until heads occurs twice. Let $X_1$ equal the number of flips up to and including the first $H$. Let $X_2$ equal the number of additional flips up to and including the sec-
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ond $H$. Let $Y = X_1 - X_2$. Find $E[Y]$ and $\text{Var}[Y]$. Hint: Don’t try to find $P_Y(y)$.

**5.7.8** $X_1$ and $X_2$ are independent identically distributed random variables with expected value $E[X]$ and variance $\text{Var}[X]$.

(a) What is $E[X_1 - X_2]$?
(b) What is $\text{Var}[X_1 - X_2]$?

**5.7.9** $X$ and $Y$ are identically distributed random variables with $E[X] = E[Y] = 0$ and covariance $\text{Cov}[X, Y] = 3$ and correlation coefficient $\rho_{X,Y} = 1/2$. For nonzero constants $a$ and $b$, $U = aX$ and $V = bY$.

(a) Find $\text{Cov}[U, V]$.
(b) Find the correlation coefficient $\rho_{U, V}$.
(c) Let $W = U + V$. For what values of $a$ and $b$ are $X$ and $W$ uncorrelated?

**5.7.10** True or False: For identically distributed random variables $Y_1$ and $Y_2$ with $E[Y_1] = E[Y_2] = 0$, $\text{Var}[Y_1 + Y_2] \geq \text{Var}[Y_1]$.

**5.7.11** $X$ and $Y$ are random variables with $E[X] = E[Y] = 0$ such that $X$ has standard deviation $\sigma_X = 2$ while $Y$ has standard deviation $\sigma_Y = 4$.

(a) For $V = X - Y$, what are the smallest and largest possible values of $\text{Var}[V]$?
(b) For $W = X - 2Y$, what are the smallest and largest possible values of $\text{Var}[W]$?

**5.7.12** Random variables $X$ and $Y$ have joint PDF

$$f_{X,Y}(x,y) = \begin{cases} 4xy & 0 \leq x \leq 1, 0 \leq y \leq 1, \\ 0 & \text{otherwise}. \end{cases}$$

(a) What are $E[X]$ and $\text{Var}[X]$?
(b) What are $E[Y]$ and $\text{Var}[Y]$?
(c) What is $\text{Cov}[X, Y]$?
(d) What is $E[X + Y]$?
(e) What is $\text{Var}[X + Y]$?

**5.7.13** Random variables $X$ and $Y$ have joint PDF

$$f_{X,Y}(x,y) = \begin{cases} 5x^2/2 & -1 \leq x \leq 1; \\ 0 & 0 \leq y \leq x^2, \\ 0 & \text{otherwise}. \end{cases}$$

Answer the following questions.

(a) What are $E[X]$ and $\text{Var}[X]$?
(b) What are $E[Y]$ and $\text{Var}[Y]$?
(c) What is $\text{Cov}[X, Y]$?
(d) What is $E[X + Y]$?
(e) What is $\text{Var}[X + Y]$?

**5.7.14** Random variables $X$ and $Y$ have joint PDF

$$f_{X,Y}(x,y) = \begin{cases} 2 & 0 \leq y \leq x \leq 1, \\ 0 & \text{otherwise}. \end{cases}$$

(a) What are $E[X]$ and $\text{Var}[X]$?
(b) What are $E[Y]$ and $\text{Var}[Y]$?
(c) What is $\text{Cov}[X, Y]$?
(d) What is $E[X + Y]$?
(e) What is $\text{Var}[X + Y]$?

**5.7.15** A transmitter sends a signal $X$ and a receiver makes the observation $Y = X + Z$, where $Z$ is a receiver noise that is independent of $X$ and $E[X] = E[Z] = 0$. Since the average power of the signal is $E[X^2]$ and the average power of the noise is $E[Z^2]$, a quality measure for the received signal is the signal-to-noise ratio

$$\Gamma = \frac{E[X^2]}{E[Z^2]}.$$ \[ \text{How is } \Gamma \text{ related to the correlation coefficient } \rho_{X,Y}? \]

**5.8.1** $X$ and $Z$ are independent random variables with $E[X] = E[Z] = 0$ and variance $\text{Var}[X] = 1$ and $\text{Var}[Z] = 16$. Let $Y = X + Z$. Find the correlation coefficient $\rho$ of $X$ and $Y$. Are $X$ and $Y$ independent?

**5.8.2** For the random variables $X$ and $Y$ in Problem 5.2.1, find

(a) The expected value of $W = Y/X$,
(b) The correlation, $r_{X,Y} = E[XY]$,
(c) The covariance, $\text{Cov}[X,Y]$,
(d) The correlation coefficient, $\rho_{X,Y}$,  
(e) The variance of $X + Y$, $\text{Var}[X + Y]$.  
(Refer to the results of Problem 5.3.1 to answer some of these questions.)

5.8.3 For the random variables $X$ and $Y$ in Problem 5.2.2 find  
(a) The expected value of $W = 2^{XY}$,  
(b) The correlation, $r_{X,Y} = \text{E}[XY]$,  
(c) The covariance, $\text{Cov}[X,Y]$,  
(d) The correlation coefficient, $\rho_{X,Y}$,  
(e) The variance of $X + Y$, $\text{Var}[X + Y]$.  
(Refer to the results of Problem 5.3.2 to answer some of these questions.)

5.8.4 Let $H$ and $B$ be the random variables in Quiz 5.3. Find $r_{H,B}$ and $\text{Cov}[H,B]$.  

5.8.5 $X$ and $Y$ are independent random variables with PDFs  
\[ f_X(x) = \begin{cases} \frac{3}{2} e^{-x/3} & x \geq 0, \\ 0 & \text{otherwise}, \end{cases} \]  
\[ f_Y(y) = \begin{cases} \frac{1}{2} e^{-y/2} & y \geq 0, \\ 0 & \text{otherwise}. \end{cases} \]

(a) Find the correlation $r_{X,Y}$.  
(b) Find the covariance $\text{Cov}[X,Y]$.  

5.8.6 The random variables $X$ and $Y$ have joint PMF  
\[ P_{X,Y}(x,y) = \begin{cases} \frac{1}{8} & (x,y) = (1,4), \\ \frac{1}{8} & (x,y) = (1,3), \\ \frac{3}{8} & (x,y) = (2,2), \\ \frac{3}{8} & (x,y) = (2,1), \\ \frac{1}{4} & (x,y) = (3,1), \\ 0 & \text{otherwise}. \end{cases} \]

Find  
(a) The expected values $\text{E}[X]$ and $\text{E}[Y]$,  
(b) The variances $\text{Var}[X]$ and $\text{Var}[Y]$.  

5.8.7 For $X$ and $Y$ with PMF $P_{X,Y}(x,y)$ given in Problem 5.8.6, let $W = \min(X,Y)$ and $V = \max(X,Y)$. Find  
(a) The expected values, $\text{E}[W]$ and $\text{E}[V]$,  
(b) The variances, $\text{Var}[W]$ and $\text{Var}[V]$,  
(c) The correlation, $r_{W,V}$,  
(d) The covariance, $\text{Cov}[W,V]$,  
(e) The correlation coefficient, $\rho_{W,V}$.  

5.8.8 Random variables $X$ and $Y$ have joint PDF  
\[ f_{X,Y}(x,y) = \begin{cases} 1/2 & -1 \leq x \leq y \leq 1, \\ 0 & \text{otherwise}. \end{cases} \]

Find $r_{X,Y}$ and $\text{E}[e^{X+Y}]$.  

5.8.9 This problem outlines a proof of Theorem 5.13.  
(a) Show that  
\[ \hat{X} - \text{E}[\hat{X}] = a(X - \text{E}[X]), \]  
\[ \hat{Y} - \text{E}[\hat{Y}] = c(Y - \text{E}[Y]). \]

(b) Use part (a) to show that  
\[ \text{Cov} [\hat{X}, \hat{Y}] = ac \text{Cov} [X, Y]. \]

(c) Show that $\text{Var}[\hat{X}] = a^2 \text{Var}[X]$ and $\text{Var}[\hat{Y}] = c^2 \text{Var}[Y]$.  
(d) Combine parts (b) and (c) to relate $\rho_{\hat{X}, \hat{Y}}$ and $\rho_{X,Y}$.  

5.8.10 Random variables $N$ and $K$ have the joint PMF  
\[ P_{N,K}(n,k) = \begin{cases} (1-p)^{n-1}p/n & k = 1, \ldots, n; \\ 0 & n = 1, 2, \ldots, \\ 0 & \text{otherwise.} \end{cases} \]

Find the marginal PMF $P_N(n)$ and the expected values $\text{E}[N]$, $\text{Var}[N]$, $\text{E}[N^2]$, $\text{E}[K]$, $\text{Var}[K]$, $\text{E}[N + K]$, $r_{N,K}$, $\text{Cov}[N,K]$.  


5.9.1 Random variables $X$ and $Y$ have joint PDF

$$f_{X,Y}(x, y) = ce^{-(x^2/8) - (y^2/18)}.$$  

What is the constant $c$? Are $X$ and $Y$ independent?

5.9.2 $X$ is the Gaussian ($\mu = 1, \sigma = 2$) random variable. $Y$ is the Gaussian ($\mu = 2, \sigma = 4$) random variable. $X$ and $Y$ are independent.

(a) What is the PDF of $V = X + Y$?

(b) What is the PDF of $W = 3X + 2Y$?

5.9.3 TRUE OR FALSE: $X_1$ and $X_2$ are bivariate Gaussian random variables. For any constant $y$, there exists a constant $a$ such that $P[X_1 + aX_2 \leq y] = 1/2$.

5.9.4 $X_1$ and $X_2$ are identically distributed Gaussian $(0, 1)$ random variables. Moreover, they are jointly Gaussian. Under what conditions are $X_1$, $X_2$, and $X_1 + X_2$ identically distributed?

5.9.5 Random variables $X$ and $Y$ have joint PDF

$$f_{X,Y}(x, y) = ce^{-(2x^2 - 4xy + 4y^2)}.$$  

(a) What are $E[X]$ and $E[Y]$?

(b) Find the correlation coefficient $\rho_{X,Y}$.

(c) What are $\text{Var}[X]$ and $\text{Var}[Y]$?

(d) What is the constant $c$?

(e) Are $X$ and $Y$ independent?

5.9.6 An archer shoots an arrow at a circular target of radius 50 cm. The arrow pierces the target at a random position $(X, Y)$, measured in centimeters from the center of the disk at position $(X, Y) = (0, 0)$. The bullseye is a solid black circle of radius 2 cm, at the center of the target. Calculate the probability $P[B]$ of the event that the archer hits the bullseye under each of the following models:

(a) $X$ and $Y$ are iid continuous uniform

(b) The PDF $f_{X,Y}(x, y)$ is uniform over the 50 cm circular target.

(c) $X$ and $Y$ are iid Gaussian ($\mu = 10$) random variables.

5.9.7 A person’s white blood cell (WBC) count $W$ (measured in thousands of cells per microliter of blood) and body temperature $T$ (in degrees Celsius) can be modeled as bivariate Gaussian random variables such that $W$ is Gaussian $(7, 2)$ and $T$ is Gaussian $(37, 1)$. To determine whether a person is sick, first the person’s temperature $T$ is measured. If $T > 38$, then the person’s WBC count is measured. If $W > 10$, the person is declared ill (event $I$).

(a) Suppose $W$ and $T$ are uncorrelated. What is $P[I]$? Hint: Draw a tree diagram for the experiment.

(b) Now suppose $W$ and $T$ have correlation coefficient $\rho_{W,T} = 1/\sqrt{2}$. Find the conditional probability $P[I|T = t]$ that a person is declared ill given that the person’s temperature is $T = t$.

5.9.8 Suppose your grade in a probability course depends on your exam scores $X_1$ and $X_2$. The professor, a fan of probability, releases exam scores in a normalized fashion such that $X_1$ and $X_2$ are iid Gaussian ($\mu = 0, \sigma = \sqrt{2}$) random variables. Your semester average is $X = 0.5(X_1 + X_2)$.

(a) You earn an $A$ grade if $X > 1$. What is $P[A]$?

(b) To improve his SIRS (Student Instructional Rating Service) score, the professor decides he should award more $A$’s. Now you get an $A$ if $\max(X_1, X_2) > 1$. What is $P[A]$ now?

(c) The professor found out he is unpopular at ratemyprofessor.com and decides to award an $A$ if either $X > 1$ or $\max(X_1, X_2) > 1$. Now what is $P[A]$?

(d) Under criticism of grade inflation from the department chair, the professor adopts a new policy. An $A$ is awarded if $\max(X_1, X_2) > 1$ and $\min(X_1, X_2) > 0$. Now what is $P[A]$?
5.9.9 Your course grade depends on two test scores: \( X_1 \) and \( X_2 \). Your score \( X_i \) on test \( i \) is Gaussian \((\mu = 74, \sigma = 16)\) random variable, independent of any other test score.

(a) With equal weighting, grades are determined by \( Y = X_1/2 + X_2/2 \). You earn an A if \( Y \geq 90 \). What is \( P[A] = P[Y \geq 90] \)?

(b) A student asks the professor to choose a weight factor \( w \), \( 0 \leq w \leq 1 \), such that
\[
Y = wX_1 + (1 - w)X_2.
\]
Find \( P[A] \) as a function of the weight \( w \). What value or values of \( w \) maximize \( P[A] = P[Y \geq 90] \)?

(c) A different student proposes that the better exam is the one that should count and that grades should be based on \( M = \max(X_1, X_2) \). In a fit of generosity, the professor agrees! Now what is \( P[A] = P[M > 90] \)?

(d) How generous was the professor? In a class of 100 students, what is the expected increase in the number of A’s awarded?

5.9.10 For any constants \( a, b, c, \) and \( d \) is
\[
f(x, y) = de^{-\left(a x^2 + b x y + c y^2\right)}
\]
a joint Gaussian PDF?

5.9.11 Show that the joint Gaussian PDF \( f_{X,Y}(x, y) \) given by Definition 5.10 satisfies
\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X,Y}(x, y) \, dx \, dy = 1.
\]
Hint: Use Equation (5.68) and the result of Problem 4.6.13.

5.9.12 Random variables \( X_1 \) and \( X_2 \) are independent identical Gaussian \((0, 1)\) random variables. Let
\[
V = X_1 - X_2, \quad W = \frac{X_1}{X_2}
\]
where
\[
\text{sgn}(x) = \begin{cases} 
1 & x > 0, \\
-1 & x \leq 0.
\end{cases}
\]

(a) Find the CDF \( F_Y(y) \) in terms of the \( \Phi(\cdot) \) function.

(b) Show that \( Y_1 \) and \( Y_2 \) are both Gaussian random variables.

(c) Are \( Y_1 \) and \( Y_2 \) bivariate Gaussian random variables?

5.10.1 Every laptop returned to a repair center is classified according to its needed repairs: (1) LCD screen, (2) motherboard, (3) keyboard, or (4) other. A random broken laptop needs a type \( i \) repair with probability \( p_i = 2^{-i}/15 \). Let \( N_i \) equal the number of type \( i \) broken laptops returned on a day in which four laptops are returned.

(a) Find the joint PMF of \( N_1, N_2, N_3, N_4 \).

(b) What is the probability that two laptops require LCD repairs?

(c) What is the probability that more laptops require motherboard repairs than keyboard repairs?

5.10.2 When ordering a personal computer, a customer can add the following features to the basic configuration: (1) additional memory, (2) flat panel display, (3) professional software, and (4) wireless modem. A random computer order has feature \( i \) with probability \( p_i = 2^{-i} \) independent of other features. In an hour in which three computers are ordered, let \( N_i \) equal the number of computers with feature \( i \).

(a) Find the joint PMF
\[
P_{N_1, N_2, N_3, N_4}(n_1, n_2, n_3, n_4).
\]

(b) What is the probability of selling a computer with no additional features?

(c) What is the probability of selling a computer with at least three additional features?
5.10.3 The random variables \( X_1, \ldots, X_n \) have the joint PDF

\[
fx_1,\ldots,x_n(x_1,\ldots,x_n) = \begin{cases} 
1 & 0 \leq x_i \leq 1; \\
i = 1,\ldots,n, & 0 \text{ otherwise.}
\end{cases}
\]

Find
(a) The joint CDF, \( F_{X_1,\ldots,X_n}(x_1,\ldots,x_n) \),
(b) \( P[\min(X_1, X_2, X_3) \leq 3/4] \).

5.10.4 Are \( N_1, N_2, N_3, N_4 \) in Problem 5.10.1 independent?

5.10.5 In a compressed data file of 10,000 bytes, each byte is equally likely to be any one of 256 possible characters \( b_0, \ldots, b_{255} \) independent of any other byte. If \( N_i \) is the number of times \( b_i \) appears in the file, find the joint PMF of \( N_0, \ldots, N_{255} \). Also, what is the joint PMF of \( N_0 \) and \( N_1 \)?

5.10.6 In Example 5.22, we derived the joint PMF of the number of pages in each of four downloads:

\[
P_{X,Y,Z}(x,y,z) = \binom{4}{x,y,z} \frac{1}{3^x} \frac{1}{2^y} \frac{1}{6^z}.
\]

(a) In a group of four downloads, what is the PMF of the number of 3-page documents?
(b) In a group of four downloads, what is the expected number of 3-page documents?
(c) Given that there are two 3-page documents in a group of four, what is the joint PMF of the number of 1-page documents and the number of 2-page documents?
(d) Given that there are two 3-page documents in a group of four, what is the expected number of 1-page documents?
(e) In a group of four downloads, what is the joint PMF of the number of 1-page documents and the number of 2-page documents?

5.10.7 \( X_1, X_2, X_3 \) are iid exponential (\( \lambda \)) random variables. Find:
(a) the PDF of \( V = \min(X_1, X_2, X_3) \),
(b) the PDF of \( W = \max(X_1, X_2, X_3) \).

5.10.8 In a race of 10 sailboats, the finishing times of all boats are iid Gaussian random variables with expected value 35 minutes and standard deviation 5 minutes.

(a) What is the probability that the winning boat will finish the race in less than 25 minutes?
(b) What is the probability that the last boat will cross the finish line in more than 50 minutes?
(c) Given this model, what is the probability that a boat will finish before it starts (negative finishing time)?

5.10.9 Random variables \( X_1, X_2, \ldots, X_n \) are iid; each \( X_j \) has CDF \( F_X(x) \) and PDF \( f_X(x) \). Consider

\[
L_n = \min(X_1, \ldots, X_n) \\
U_n = \max(X_1, \ldots, X_n).
\]

In terms of \( F_X(x) \) and/or \( f_X(x) \):
(a) Find the CDF \( F_{L_n}(u) \).
(b) Find the CDF \( F_{U_n}(l) \).
(c) Find the joint CDF \( F_{L_n,U_n}(l,u) \).

5.10.10 Suppose you have \( n \) suitcases and suitcase \( i \) holds \( X_i \) dollars where \( X_1, X_2, \ldots, X_n \) are iid continuous uniform \((0, m)\) random variables. (Think of a number like one million for the symbol \( m \).) Unfortunately, you don’t know \( X_i \) until you open suitcase \( i \).

Suppose you can open the suitcases one by one, starting with suitcase \( n \) and going down to suitcase 1. After opening suitcase \( i \), you can either accept or reject \( X_i \) dollars. If you accept suitcase \( i \), the game ends. If you reject, then you get to choose only from the still unopened suitcases.

What should you do? Perhaps it is not so obvious? In fact, you can decide before the game on a policy, a set of rules to follow. We will specify a policy by a vector \((\tau_1, \ldots, \tau_n)\) of threshold parameters.
• After opening suitcase $i$, you accept the amount $X_i$ if $X_i \geq \tau_i$.
• Otherwise, you reject suitcase $i$ and open suitcase $i - 1$.
• If you have rejected suitcases $n$ down through 2, then you must accept the amount $X_1$ in suitcase 1. Thus the threshold $\tau_1 = 0$ since you never reject the amount in the last suitcase.

(a) Suppose you reject suitcases $n$ through $i + 1$, but then you accept suitcase $i$. Find $E[X_i|X_i \geq \tau_i]$.
(b) Let $W_k$ denote your reward given that there are $k$ unopened suitcases remaining. What is $E[W_1]$?
(c) As a function of $\tau_k$, find a recursive relationship for $E[W_k]$ in terms of $\tau_k$ and $E[W_{k-1}]$.
(d) For $n = 4$ suitcases, find the policy $(\tau_1^*, \ldots, \tau_4^*)$, that maximizes $E[W_4]$.

5.10.1.1 Given the set $\{U_1, \ldots, U_n\}$ of iid uniform $(0, T)$ random variables, we define

$$X_k = \text{small}_k(U_1, \ldots, U_n)$$

as the $k$th "smallest" element of the set. That is, $X_1$ is the minimum element, $X_2$ is the second smallest, and so on, up to $X_n$, which is the maximum element of $\{U_1, \ldots, U_n\}$. Note that $X_1, \ldots, X_n$ are known as the order statistics of $U_1, \ldots, U_n$.

Prove that

$$f_{X_1, \ldots, X_n}(x_1, \ldots, x_n) = \begin{cases} n!/T^n & 0 \leq x_1 < \cdots < x_n \leq T, \\ 0 & \text{otherwise.} \end{cases}$$

5.11.1 For random variables $X$ and $Y$ in Example 5.26, use MATLAB to generate a list of the form

$$\begin{align*}
x_1 & \quad y_1 & \quad P_{X,Y}(x_1, y_1) \\
x_2 & \quad y_2 & \quad P_{X,Y}(x_2, y_2) \\
& \quad \vdots & \quad \vdots 
\end{align*}$$

that includes all possible pairs $(x, y)$.

5.11.2 For random variables $X$ and $Y$ in Example 5.26, use MATLAB to calculate $E[X]$, $E[XY]$, the correlation $E[XY^\prime]$, and the covariance $\text{Cov}[X, Y]$.

5.11.3 You generate random variable $W = W$ by typing $W=\text{sum}(4*\text{randn}(1,2))$ in a MATLAB Command window. What is $\text{Var}[W]$?

5.11.4 Write $\text{trianglecdfplot.m}$, a script that graphs $F_{X,Y}(x,y)$ of Figure 5.4.

5.11.5 Problem 5.2.6 extended Example 5.3 to a test of $n$ circuits and identified the joint PDF of $X$, the number of acceptable circuits, and $Y$, the number of successful tests before the first reject. Write a MATLAB function

$$[S_X, S_Y, P_{XY}]=\text{circuits}(n, p)$$

that generates the sample space grid for the $n$ circuit test. Check your answer against Equation (5.11) for the $p = 0.9$ and $n = 2$ case. For $p = 0.9$ and $n = 50$, calculate the correlation coefficient $\rho_{X,Y}$.
Problems

6.1.1 Random variables \( X \) and \( Y \) have joint PMF

\[
P_{X,Y}(x,y) = \begin{cases} \frac{|x + y|}{14} & x = -2, 0, 2; \\ y = -1, 0, 1, \\ 0 & \text{otherwise.} \end{cases}
\]

Find the PMF of \( W = X - Y \).

6.1.2 For random variables \( X \) and \( Y \) in Problem 6.1.1, find the PMF of \( W = X + 2Y \).

6.1.3 \( N \) is a binomial \( (n = 100, p = 0.4) \) random variable. \( M \) is a binomial \( (n = 50, p = 0.4) \) random variable. Given that \( M \) and \( N \) are independent, what is the PMF of \( L = M + N \)?

6.1.4 Let \( X \) and \( Y \) be discrete random variables with joint PMF \( P_{X,Y}(x,y) \) that is zero except when \( x \) and \( y \) are integers. Let \( W = X + Y \) and show that the PMF of \( W \) satisfies

\[
P_W(w) = \sum_{x=-\infty}^{\infty} P_{X,Y}(x, w-x).
\]

6.1.5 Let \( X \) and \( Y \) be discrete random variables with joint PMF

\[
P_{X,Y}(x,y) = \begin{cases} 0.01 & x = 1, 2, \ldots, 10, \\ y = 1, 2, \ldots, 10, \\ 0 & \text{otherwise.} \end{cases}
\]

What is the PMF of \( W = \min(X, Y) \)?

6.1.6 For random variables \( X \) and \( Y \) in Problem 6.1.5, what is the PMF of \( V = \max(X, Y) \)?

6.2.1 The voltage \( X \) across a 1 \( \Omega \) resistor is a uniform random variable with parameters 0 and 1. The instantaneous power is \( Y = X^2 \). Find the CDF \( F_Y(y) \) and the PDF \( f_Y(y) \) of \( Y \).

6.2.2 \( X \) is the Gaussian \((0, 1)\) random variable. Find the CDF of \( Y = |X| \) and its expected value \( E[Y] \).

6.2.3 In a 50 km Tour de France time trial, a rider’s time \( T \), measured in minutes, is the continuous uniform \((60, 75)\) random variable. Let \( V = 3000/T \) denote the rider’s speed over the course in km/hr. Find the PDF of \( V \).

6.2.4 In the presence of a headwind of normalized intensity \( W \), your speed on your bike is \( V = g(W) = 20 - 10W^{1/3} \) mi/hr. The wind intensity \( W \) is the continuous uniform \((-1, 1)\) random variable. (Note: If \( W \) is negative, then the headwind is actually a tailwind.) Find the PDF \( f_V(v) \).

6.2.5 If \( X \) has an exponential (\( \lambda \)) PDF, what is the PDF of \( W = X^2 \)?

6.2.6 Let \( X \) denote the position of the pointer after a spin on a wheel of circumference 1. For that same spin, let \( Y \) denote the area within the arc defined by the stopping position of the pointer:

(a) What is the relationship between \( X \) and \( Y \)?

(b) What is \( F_Y(y) \)?

(c) What is \( f_Y(y) \)?

(d) What is \( E[Y] \)?

6.2.7 \( U \) is the uniform \((0, 1)\) random variable and \( X = -\ln(1-U) \).

(a) What is \( F_X(x) \)?

(b) What is \( f_X(x) \)?

(c) What is \( E[X] \)?
6.2.8 \( X \) is the uniform \((0, 1)\) random variable. Find a function \( g(x) \) such that the PDF of \( Y = g(X) \) is

\[
 f_Y(y) = \begin{cases} 
 3y^2 & 0 \leq y \leq 1, \\
 0 & \text{otherwise}.
\end{cases}
\]

6.2.9 An amplifier circuit has power consumption \( Y \) that grows nonlinearly with the input signal voltage \( X \). When the input signal is \( X \) volts, the instantaneous power consumed by the amplifier is \( Y = 20 + 15X^2 \) Watts. The input signal \( X \) is the continuous uniform \((-1, 1)\) random variable. Find the PDF \( f_Y(y) \).

6.2.10 Use Theorem 6.2 to prove Theorem 6.3.

6.2.11 For the uniform \((0, 1)\) random variable \( U \), find the CDF and PDF of \( Y = a + (b - a)U \) with \( a < b \). Show that \( Y \) is the uniform \((a, b)\) random variable.

6.2.12 Theorem 6.5 required the inverse CDF \( F^{-1}(u) \) to exist for \( 0 < u < 1 \). Why was it not necessary that \( F^{-1}(u) \) exist at either \( u = 0 \) or \( u = 1 \)?

6.2.13 \( X \) is a continuous random variable. \( Y = aX + b \), where \( a, b \neq 0 \). Prove that

\[
 f_Y(y) = \frac{f_X((y - b)/a)}{|a|}.
\]

Hint: Consider the cases \( a < 0 \) and \( a > 0 \) separately.

6.2.14 Let continuous random variable \( X \) have a CDF \( F(x) \) such that \( F^{-1}(u) \) exists for all \( u \) in \([0, 1]\). Show that \( U = F(X) \) is the uniform \((0, 1)\) random variable. Hint: \( U \) is a random variable such that when \( X = x' \), \( U = F(x') \). That is, we evaluate the CDF of \( X \) at the observed value of \( X \).

6.3.1 \( X \) has CDF

\[
 F_X(x) = \begin{cases} 
 0 & x < -1, \\
 x/3 + 1/3 & -1 \leq x < 0, \\
 x/3 + 2/3 & 0 \leq x < 1, \\
 1 & 1 \leq x.
\end{cases}
\]

\( Y = g(X) \) where

\[
 g(X) = \begin{cases} 
 0 & X < 0, \\
 100 & X \geq 0.
\end{cases}
\]

(a) What is \( F_Y(y) \)?
(b) What is \( f_Y(y) \)?
(c) What is \( E[Y] \)?

6.3.2 In a 50 km cycling time trial, a rider’s exact time \( T \), measured in minutes, is the continuous uniform \((50, 60)\) random variable. However, a rider’s recorded time \( R \) in seconds is obtained by rounding up \( T \) to next whole second. That is, if \( T \) is 50 minutes, 27.001 seconds, then \( R = 3028 \) seconds. On the other hand, if \( T \) is exactly 50 minutes 27 seconds, then \( R = 3027 \). What is the PMF of \( R \)?

6.3.3 The voltage \( V \) at the output of a microphone is the continuous uniform \((-1, 1)\) random variable. The microphone voltage is processed by a clipping rectifier with output

\[
 L = \begin{cases} 
 |V| & |V| \leq 0.5, \\
 0.5 & \text{otherwise}.
\end{cases}
\]

(a) What is \( P[L = 0.5] \)?
(b) What is \( F_L(l) \)?
(c) What is \( E[L] \)?

6.3.4 \( U \) is the uniform random variable with parameters 0 and 2. The random variable \( W \) is the output of the clipper:

\[
 W = g(U) = \begin{cases} 
 U & U \leq 1, \\
 1 & U > 1.
\end{cases}
\]

Find the CDF \( F_W(w) \), the PDF \( f_W(w) \), and the expected value \( E[W] \).
6.3.5 \( X \) is a random variable with CDF \( F_X(x) \). Let \( Y = g(X) \) where

\[
  g(x) = \begin{cases} 
    10 & x < 0, \\
    -10 & x \geq 0. 
  \end{cases}
\]

Express \( F_Y(y) \) in terms of \( F_X(x) \).

6.3.6 Suppose that a cellular phone costs $30 per month with 300 minutes of use included and that each additional minute of use costs $0.50. The number of minutes you use the phone in a month is an exponential random variable \( T \) with with expected value \( E[T] = 200 \) minutes. The telephone company charges you for exactly how many minutes you use without any rounding of fractional minutes. Let \( C \) denote the cost in dollars of one month of service.

(a) What is \( P[C = 30] \)?
(b) What is the PDF of \( C \)?
(c) What is \( E[C] \)?

6.3.7 The input voltage to a rectifier is the continuous uniform \((0, 1)\) random variable \( U \). The rectifier output is a random variable \( W \) defined by

\[
  W = g(U) = \begin{cases} 
    0 & U < 0, \\
    U & U \geq 0. 
  \end{cases}
\]

Find the CDF \( F_W(w) \) and the expected value \( E[W] \).

6.3.8 Random variable \( X \) has PDF

\[
  f_X(x) = \begin{cases} 
    x/2 & 0 \leq x \leq 2, \\
    0 & \text{otherwise}. 
  \end{cases}
\]

\( X \) is processed by a clipping circuit with output

\[
  Y = \begin{cases} 
    0.5 & X \leq 1, \\
    X & X > 1. 
  \end{cases}
\]

(a) What is \( P[Y = 0.5] \)?
(b) Find the CDF \( F_Y(y) \).

6.3.9 Given an input voltage \( V \), the output voltage \( W \) is

\[
  W = \begin{cases} 
    0 & V \leq 0, \\
    V & 0 < V < 10, \\
    10 & V \geq 10. 
  \end{cases}
\]

Suppose the input \( V \) is the continuous uniform \((-15, 15)\) random variable. Find the PDF of \( W \).

6.3.10 The current \( X \) across a resistor is the continuous uniform \((-2, 2)\) random variable. The power dissipated in the resistor is \( Y = 9X^2 \) Watts.

(a) Find the CDF and PDF of \( Y \).
(b) A power measurement circuit is range-limited so that its output is

\[
  W = \begin{cases} 
    Y & Y < 16, \\
    16 & \text{otherwise}. 
  \end{cases}
\]

Find the PDF of \( W \).

6.3.11 A defective voltmeter measures small voltages as zero. In particular, when the input voltage is \( V \), the measured voltage is

\[
  W = \begin{cases} 
    0 & |V| < 0.6, \\
    V & \text{otherwise}. 
  \end{cases}
\]

If \( V \) is the continuous uniform \((-5, 5)\) random variable, what is the PDF of \( W \)?

6.3.12 \( X \) is the continuous uniform \((-3, 3)\) random variable. When \( X \) is passed through a limiter, the output is the discrete random variable

\[
  \hat{X} = g(X) = \begin{cases} 
    -c & X < 0, \\
    c & X \geq 0, 
  \end{cases}
\]

where \( c \) is an unspecified positive constant.

(a) What is the PMF \( P_{\hat{X}}(x) \) of \( \hat{X} \)?
(b) When the limiter input is \( X \), the distortion \( D \) between the input \( X \) and the limiter output \( \hat{X} \) is

\[
  D = d(X) - (y - d(X))^2
\]
In terms of $c$, find the expected distortion $E[D] = E[d(X)]$. What value of $c$ minimizes $E[D]$?

(c) $Y$ is a Gaussian random variable with the same expected value and variance as $X$. What is the PDF of $Y$?

(d) Suppose $Y$ is passed through the limiter yielding the output $\hat{Y} = g(Y)$. The distortion $D$ between the input $Y$ and the limiter output $\hat{Y}$ is

$$D = d(Y) = (Y - g(Y))^2.$$ 

In terms of $c$, find the expected distortion $E[D] = E[d(Y)]$. What value of $c$ minimizes $E[D]$?

6.3.13 In this problem we prove a generalization of Theorem 6.5. Given a random variable $X$ with CDF $F_X(x)$, define

$$\bar{F}(u) = \min\{x|F_X(x) \geq u\}.$$ 

This problem proves that for a continuous uniform $(0, 1)$ random variable $U$, $\bar{X} = \bar{F}(U)$ has CDF $F_{\bar{X}}(x) = F_X(x)$.

(a) Show that when $F_X(x)$ is a continuous, strictly increasing function (i.e., $X$ is not mixed, $F_X(x)$ has no jump discontinuities, and $F_X(x)$ has no "flat" intervals $(a, b)$ where $F_X(x) = c$ for $a \leq x \leq b$), then $\bar{F}(u) = F_X^{-1}(u)$ for $0 < u < 1$.

(b) Show that if $F_X(x)$ has a jump at $x = x_0$, then $\bar{F}(u) = x_0$ for all $u$ in the interval $F_X(x_0^-) \leq u \leq F_X(x_0^+)$.

(c) Prove that $\bar{X} = \bar{F}(U)$ has CDF $F_{\bar{X}}(x) = F_X(x)$.

6.4.1 Random variables $X$ and $Y$ have joint PDF

$$f_{X,Y}(x,y) = \begin{cases} 2xy^2 & 0 \leq x, y \leq 1, \\ 0 & \text{otherwise.} \end{cases}$$

Let $V = \max(X,Y)$. Find the CDF and

6.4.2 For random variables $X$ and $Y$ in Problem 6.4.1, find the CDF and PDF of $W = \min(X,Y)$.

6.4.3 $X$ and $Y$ have joint PDF

$$f_{X,Y}(x,y) = \begin{cases} 2 & x \geq 0, y \geq 0, x + y \leq 1, \\ 0 & \text{otherwise.} \end{cases}$$

(a) Are $X$ and $Y$ independent?

(b) Let $U = \min(X,Y)$. Find the CDF and PDF of $U$.

(c) Let $V = \max(X,Y)$. Find the CDF and PDF of $V$.

6.4.4 Random variables $X$ and $Y$ have joint PDF

$$f_{X,Y}(x,y) = \begin{cases} x + y & 0 \leq x, y \leq 1, \\ 0 & \text{otherwise.} \end{cases}$$

Let $W = \max(X,Y)$.

(a) What is $S_W$, the range of $W$?

(b) Find $F_W(w)$ and $f_W(w)$.

6.4.5 Random variables $X$ and $Y$ have joint PDF

$$f_{X,Y}(x,y) = \begin{cases} 6y & 0 \leq y \leq x \leq 1, \\ 0 & \text{otherwise.} \end{cases}$$

Let $W = Y - X$.

(a) What is $S_W$, the range of $W$?

(b) Find $F_W(w)$ and $f_W(w)$.

6.4.6 Random variables $X$ and $Y$ have joint PDF

$$f_{X,Y}(x,y) = \begin{cases} 2 & 0 \leq y \leq x \leq 1, \\ 0 & \text{otherwise.} \end{cases}$$

Let $W = Y/X$.

(a) What is $S_W$, the range of $W$?

(b) Find $F_W(w)$, $f_W(w)$, and $E[W]$.

6.4.7 Random variables $X$ and $Y$ have joint PDF

$$f_{X,Y}(x,y) = \begin{cases} \frac{2}{2} & 0 \leq y \leq x \leq 1, \\ \end{cases}$$
Let $W = X/Y$.
(a) What is $S_W$, the range of $W$?
(b) Find $F_W(w)$, $f_W(w)$, and $E[W]$.

**6.4.8** In a simple model of a cellular telephone system, a portable telephone is equally likely to be found anywhere in a circular cell of radius 4 km. (See Problem 5.5.4.) Find the CDF $F_R(r)$ and PDF $f_R(r)$ of $R$, the distance (in km) between the telephone and the base station at the center of the cell.

**6.4.9** $X$ and $Y$ are independent identically distributed Gaussian $(0, 1)$ random variables. Find the CDF of $W = X^2 + Y^2$.

**6.4.10** $X$ is the exponential $(2)$ random variable and $Z$ is the Bernoulli $(1/2)$ random variable that is independent of $X$. Find the PDF of $Y = ZX$.

**6.4.11** $X$ is the Gaussian $(0, 1)$ random variable and $Z$, independent of $X$, has PMF

$$P_Z(z) = \begin{cases} 1 - p & z = -1, \\ p & z = 1. \end{cases}$$

Find the PDF of $Y = ZX$.

**6.4.12** You are waiting on the platform of the first stop of a Manhattan subway line. You could ride either a local or express train to your destination, which is the last stop on the line. The waiting time $X$ for the next express train is the exponential random variable with $E[X] = 10$ minutes. The waiting time $Y$ for the next local train is the exponential random variable with $E[Y] = 5$ minutes. Although the arrival times $X$ and $Y$ of the trains are random and independent, the trains' travel times are deterministic; the local train travels from first stop to last stop in exactly 15 minutes while the express travels from first to last stop in exactly 5 minutes.
(a) What is the joint PDF $f_{X,Y}(x,y)$?
(b) Find $P[L]$ that the local train arrives first at the platform?
(c) Suppose you board the first train that arrives. Find the PDF of your waiting
(d) The time until the first train (express or local) reaches final stop is $T = \min(X + 5, Y + 15)$. Find $f_T(t)$.
(e) Suppose the local train does arrive first at your platform. Should you board the local train? Justify your answer. (There may be more than one correct answer.)

**6.4.13** For a constant $a > 0$, random variables $X$ and $Y$ have joint PDF

$$f_{X,Y}(x, y) = \begin{cases} \frac{1}{a^2} & 0 \leq x, y \leq a, \\ 0 & \text{otherwise}. \end{cases}$$

Find the CDF and PDF of random variable

$$W = \max \left( \frac{X}{Y}, \frac{Y}{X} \right).$$

Hint: Is it possible to observe $W < 1$?

**6.4.14** The joint PDF of $X$ and $Y$ is

$$f_{X,Y}(x, y) = \begin{cases} \lambda^2 e^{-\lambda y} & 0 \leq x < y, \\ 0 & \text{otherwise}. \end{cases}$$

What is the PDF of $W = Y - X$?

**6.4.15** Consider random variables $X$, $Y$, and $W$ from Problem 6.4.14.
(a) Are $W$ and $X$ independent?
(b) Are $W$ and $Y$ independent?

**6.4.16** $X$ and $Y$ are independent random variables with CDFs $F_X(x)$ and $F_Y(y)$. Let $U = \min(X, Y)$ and $V = \max(X, Y)$.
(a) What is $F_{U,V}(u, v)$?
(b) What is $f_{U,V}(u, v)$?

Hint: To find the joint CDF, let $A = \{U \leq u\}$ and $B = \{V \leq v\}$ and note that $P[AB] = P[B] - P[A^cB]$.

**5.5.1** Let $X$ and $Y$ be independent discrete random variables such that $P_X(k) = P_X(k) = 0$ for all non-integer $k$. Show that the PMF of $W = X + Y$ satisfies

$$P_W(w) = \sum_{k=-\infty}^{\infty} P_X(k) P_Y(w - k).$$
6.5.2 \( X \) and \( Y \) have joint PDF
\[
f_{X,Y}(x, y) = \begin{cases} 
2 & x \geq 0, y \geq 0, x + y \leq 1, \\
0 & \text{otherwise.}
\end{cases}
\]
Find the PDF of \( W = X + Y \).

6.5.3 \( X \) and \( Y \) have joint PDF
\[
f_{X,Y}(x, y) = \begin{cases} 
2 & 0 \leq x \leq y \leq 1, \\
0 & \text{otherwise.}
\end{cases}
\]

6.5.4 \( X \) and \( Y \) have joint PDF
\[
f_{X,Y}(x, y) = \begin{cases} 
1 & 0 \leq x \leq 1, 0 \leq y \leq 1, \\
0 & \text{otherwise.}
\end{cases}
\]

6.5.5 Random variables \( X \) and \( Y \) are independent exponential random variables with expected values \( E[X] = 1/\lambda \) and \( E[Y] = 1/\mu \). If \( \mu \neq \lambda \), what is the PDF of \( W = X + Y \)? If \( \mu = \lambda \), what is \( f_W(w) \)?

6.5.6 Random variables \( X \) and \( Y \) have joint PDF
\[
f_{X,Y}(x, y) = \begin{cases} 
8xy & 0 \leq y \leq x \leq 1, \\
0 & \text{otherwise.}
\end{cases}
\]
What is the PDF of \( W = X + Y \)?

6.5.7 Continuous random variables \( X \) and \( Y \) have joint PDF \( f_{X,Y}(x, y) \). Show that \( W = X - Y \) has PDF
\[
f_W(w) = \int_{-\infty}^{\infty} f_{X,Y}(y + w, y) \, dy.
\]
Use a variable substitution to show
\[
f_W(w) = \int_{-\infty}^{\infty} f_{X,Y}(x, x - w) \, dx.
\]

6.5.8 In this problem we show directly that the sum of independent Poisson random variables is Poisson. Let \( J \) and \( K \) be independent Poisson random variables with expected values \( \alpha \) and \( \beta \), respectively, and show that \( N = J + K \) is a Poisson random variable with expected value \( \alpha + \beta \). Hint: Show that
\[
P_N(n) = \sum_{m=0}^{n} P_K(m) P_J(n - m),
\]
and then simplify the summation by extracting the sum of a binomial PMF over all possible values.

6.6.1 Use \( \text{icdfv.m} \) to write a function \( \text{w=urv1}(m) \) that generates \( m \) samples of random variable \( W \) from Problem 4.2.4. Note that \( F_W^{-1}(u) \) does not exist for \( u = 1/4 \); however, you must define a function \( \text{icdfv}(u) \) that returns a value for \( \text{icdfv}(0.25) \). Does it matter what value you return for \( u=0.25 \)?

6.6.2 Write a MATLAB function \( \text{w=urv}(m) \) that generates \( m \) samples of random variable \( U \) defined in Problem 4.4.7.

6.6.3 For random variable \( W \) of Example 6.10, we can generate random samples in two different ways:

1. Generate samples of \( X \) and \( Y \) and calculate \( W = Y/X \).
2. Find the CDF \( F_W(w) \) and generate samples using Theorem 6.5.

Write MATLAB functions \( \text{w=urv1}(m) \) and \( \text{w=urv2}(m) \) to implement these methods. Does one method run much faster? If so, why? (Use \text{cputime} to make comparisons.)

6.6.4 Write a function \( \text{y=deltarpv}(m) \) that returns \( m \) samples of the random variable \( X \) with PDF
\[
F_X(x) = \begin{cases} 
0 & x < -1, \\
(x + 1)/4 & -1 \leq x < 1, \\
1 & x \geq 1.
\end{cases}
\]
Since \( F_X^{-1}(u) \) is not defined for \( 1/2 \leq u < 1 \), use the result of Problem 6.3.13.
Theorem 5.14 states that for any pair of random variables, $|\rho_{X,Y}| < 1$. Introducing this inequality to the formulas for conditional variance in Theorem 7.15 and Theorem 7.16 leads to the following inequalities:

\begin{align*}
\text{Var} [Y|X = x] &= \sigma_Y^2 (1 - \rho_{X,Y}^2) \leq \sigma_Y^2, \\
\text{Var} [X|Y = y] &= \sigma_X^2 (1 - \rho_{X,Y}^2) \leq \sigma_X^2.
\end{align*}  

(7.71)  

(7.72)

These formulas state that for $\rho_{X,Y} \neq 0$, learning the value of one of the random variables leads to a model of the other random variable with reduced variance. This suggests that learning the value of $Y$ reduces our uncertainty regarding $X$.

**Quiz 7.6**

Let $X$ and $Y$ be jointly Gaussian $(0, 1)$ random variables with correlation coefficient $1/2$. What is the conditional PDF of $X$ given $Y = 2$? What are the conditional expected value and conditional variance $E[X|Y = 2]$ and $\text{Var}[X|Y = 2]$?

### 7.7 MATLAB

To generate sample values of random variables $X$ and $Y$, use $P_X(x)$ or $f_X(x)$ to generate sample values of $X$. Then for each sample value $x_i$, use $P_Y|X(y|x_i)$ or $f_Y|X(y|x_i)$ to get a sample value of $Y$.

MATLAB provides the `find` function to identify conditions. We use the `find` function to calculate conditional PMFs for finite random variables.

**Example 7.21**

Repeating Example 7.3, find the conditional PMF for the length $X$ of a video given event $L$ that the video is long with $X \geq 5$ minutes.

```matlab
sx=(1:8)';
px=[0.15*ones(4,1); 0.1*ones(4,1)];
sxL=unique(find(sx>=5));
pL=sum(finitepmf(sx,px,sxL));
pXL=finitepmf(px,sx,sxL)/pL;
```

With random variable $X$ defined by $sx$ and $px$ as in Example 3.43, this code solves this problem. The vector $sxL$ identifies the event $L$, $pL$ is the probability $P[L]$, and $pxL$ is the vector of probabilities $P_{X|L}(x_i)$ for each $x_i \in L$.

The conditional PMF and PDF can also be used in MATLAB to simplify the generation of sample pairs $(X, Y)$. For example, when $X$ and $Y$ have the joint PDF $f_{X,Y}(x,y)$, a basic approach is to generate sample values $x_1, \ldots, x_m$ for $X$ using the marginal PDF $f_X(x)$. Then for each sample $x_i$, we generate $y_i$ using the conditional PDF $f_{Y|X}(y|x_i)$. MATLAB can do this efficiently provided the samples $y_1, \ldots, y_m$ can be generated from $x_1, \ldots, x_m$ using vector-processing techniques, as
Example 7.22

Write a function \( xy = x y \text{trian} \text{g} \text{ler} \text{v} \text{m} \) that generates \( m \) sample pairs \((X, Y)\) in Example 7.16.

In Example 7.16, we found that

\[
fx(x) = \begin{cases} 
2x & 0 \leq x \leq 1, \\
0 & \text{otherwise}, 
\end{cases} \quad f_{Y|X}(y|x) = \begin{cases} 
1/x & 0 \leq y \leq x, \\
0 & \text{otherwise}. 
\end{cases} \quad (7.73)
\]

Function

```matlab
function xy = xytetrianglev(m);
x = sqrt(rand(m,1));
y = x .* rand(m,1);
xy = [x y];
```

For \( 0 \leq x \leq 1 \), we have that \( F_X(x) = x^2 \). Using Theorem 6.5 to generate sample values of \( X \), we define \( u = F_X(x) = x^2 \). Then, for \( 0 < u < 1 \), \( x = \sqrt{u} \). By Theorem 6.5, if \( U \) is uniform \((0, 1)\), then \( \sqrt{U} \) has PDF \( f_X(x) \). Next, we observe that given \( X = x_i \), \( Y \) is the uniform \((0, x_i)\) random variable. Given another uniform \((0, 1)\) random variable \( U_i \), Theorem 6.3(a) states that \( Y_i = x_i U_i \) is the uniform \((0, x_i)\) random variable. We implement these ideas in the function `xytrianglev.m`.

Quiz 7.7

For random variables \( X \) and \( Y \) with joint PMF \( P_{X,Y}(x, y) \) given in Example 7.11, write a MATLAB function `xydtrianglev(m)` that generates \( m \) sample pairs.

Problems

7.1.1 ★ Random variable \( X \) has CDF

\[
F_X(x) = \begin{cases} 
0 & x < -3, \\
0.4 & -3 \leq x < 5, \\
0.8 & 5 \leq x < 7, \\
1 & x \geq 7. 
\end{cases}
\]

Find the conditional CDF \( F_{X|X>0}(x) \) and PMF \( P_{X|X>0}(x) \).

7.1.2 ★ \( X \) is the discrete uniform \((0, 5)\) random variable. What is \( E[X|X \geq E[X]] \)?

7.1.3 ★ \( X \) has PMF

\[
P_X(x) = \binom{4}{x} (1/2)^4.
\]

7.1.4 ★ In a youth basketball league, a player is fouled in the act of shooting a layup. There is a probability \( q = 0.2 \) that the layup is good, scoring 2 points. If the layup is good, the player is also awarded 1 free throw, giving the player a chance at a three-point play. If the layup is missed, then (because of the foul) the player is still awarded one point automatically and is also awarded one free throw, enabling a chance to score two points in total. The player makes a free throw with probability \( p = 1/2 \).

(a) What is the PMF of \( X \), the number of points scored by the player?

(b) Find the conditional PMF \( P_{X|T}(x) \) of \( X \) given event \( T \) that the free throw is made.
7.1.5 Every day you consider going jogging. Before each mile, including the first, you will quit with probability $q$, independent of the number of miles you have already run. However, you are sufficiently decisive that you never run a fraction of a mile. Also, we say you have run a marathon whenever you run at least 26 miles.

(a) Let $M$ equal the number of miles that you run on an arbitrary day. Find the PMF $P_M(m)$.

(b) Let $r$ be the probability that you run a marathon on an arbitrary day. Find $r$.

(c) Let $J$ be the number of days in one year (not a leap year) in which you run a marathon. Find the PMF $P_J(j)$. This answer may be expressed in terms of $r$ found in part (b).

(d) Define $K = M - 26$. Let $A$ be the event that you have run a marathon. Find $P_{K/A}(k)$.

7.1.6 A random ECE student has height $X$ in inches given by the PDF

$$f_X(x) = \frac{4e^{-(x-70)^2/8} + e^{-(x-65)^2/8}}{5\sqrt{8\pi}}.$$  

(a) Sketch $f_X(x)$ over the interval $60 \leq x \leq 75$. (For purposes of sketching, note that $\sqrt{8\pi} \approx 5$.)

(b) Find the probability that a random ECE student is less than 5 feet 8 inches tall.

(c) Use conditional PDFs to explain why $f_X(x)$ might be a reasonable model for ECE students.

7.1.7 A test for diabetes is a measurement $X$ of a person’s blood sugar level following an overnight fast. For a healthy person, a blood sugar level $X$ in the range of $70 - 110$ mg/dl is considered normal. When a measurement $X$ is used as a test for diabetes, the result is called positive (event $T^+$) if $X \geq 140$; the test is negative (event $T^-$) if $X \leq 110$, and the test is ambiguous (event $T^0$).

Given that a person is healthy (event $H$), a blood sugar measurement $X$ is the Gaussian $(90, 20)$ random variable. Given that a person has diabetes, (event $D$), $X$ is the Gaussian $(60, 40)$ random variable. A randomly chosen person is healthy with probability $P[H] = 0.9$ or has diabetes with probability $P[D] = 0.1$.

(a) What is the conditional PDF $f_{X|H}(x)$?

(b) Calculate the conditional probabilities $P[T^+|H]$, and $P[T^-|H]$.

(c) Find $P[H|T^-]$, the conditional probability that a person is healthy given the event of a negative test.

(d) When a person has an ambiguous test result ($T^0$), the test is repeated, possibly many times, until either a positive $T^+$ or negative $T^-$ result is obtained. Let $N$ denote the number of times the test is given. Assuming that for a given person the result of each test is independent of the result of all other tests, find the conditional PMF of $N$ given event $H$ that a person is healthy. Note that $N = 1$ if the person has a positive $T^+$ or negative $T^-$ result on the first test.

7.1.8 For the quantizer of Example 7.6, the difference $Z = X - Y$ is the quantization error or quantization “noise.” As in Example 7.6, assume that $X$ has a uniform $(-r/2, r/2)$ PDF.

(a) Given event $B_i$ that $Y = y_i = \Delta/2 + i\Delta$ and $X$ is in the $i$th quantization interval, find the conditional PDF of $Z$.

(b) Show that $Z$ is a uniform random variable. Find the PDF, the expected value, and the variance of $Z$.

7.1.9 For the quantizer of Example 7.6, we showed in Problem 7.1.8 that the quantization noise $Z$ is a uniform random variable. If $X$ is not uniform, show that $Z$ is nonuniform by calculating the PDF of $Z$ for a simple example.

7.2.1 $X$ is the binomial $(5, 1/2)$ random
tion \( B = \{X \geq \mu x\} \). What are \( E[X|B] \) and \( \text{Var}[X|B] \)?

7.2.2 Random variable \( X \) has CDF

\[
F_X(x) = \begin{cases} 
0 & x < -1, \\
0.2 & -1 \leq x < 0, \\
0.7 & 0 \leq x < 1, \\
1 & x \geq 1.
\end{cases}
\]

Given \( B = \{|X| > 0\} \), find \( P_{X|B}(x) \). What are \( E[X|B] \) and \( \text{Var}[X|B] \)?

7.2.3 \( X \) is the continuous uniform \((-5, 5)\) random variable. Given the event \( B = \{|X| \leq 3\} \), find the

(a) conditional PDF, \( f_{X|B}(x) \),
(b) conditional expected value, \( E[X|B] \),
(c) conditional variance, \( \text{Var}[X|B] \).

7.2.4 \( Y \) is the exponential (0.2) random variable. Given \( A = \{Y < 2\} \), find:

(a) \( f_{Y|A}(y) \),
(b) \( E[Y|A] \).

7.2.5 For the experiment of spinning the pointer three times and observing the maximum pointer position, Example 4.5, find the conditional PDF given the event \( R \) that the maximum position is on the right side of the circle. What are the conditional expected value and the conditional variance?

7.2.6 The number of pages \( X \) in a document has PMF

\[
P_X(x) = \begin{cases} 
0.15 & x = 1, 2, 3, 4, \\
0.1 & x = 5, 6, 7, 8, \\
0 & \text{otherwise}.
\end{cases}
\]

A firm sends all documents with an even number of pages to printer \( A \) and all documents with an odd number of pages to printer \( B \).

(a) Find the conditional PMF of the length \( X \) of a document, given the document was sent to \( A \). What are the conditional expected length and standard

(b) Find the conditional PMF of the length \( X \) of a document, given the document was sent to \( B \) and had no more than six pages. What are the conditional expected length and standard deviation?

7.2.7 4 Select integrated circuits, test them in sequence until you find the first failure, and then stop. Let \( N \) be the number of tests. All tests are independent, with probability of failure \( p = 0.1 \). Consider the condition \( B = \{N \geq 20\} \).

(a) Find the PMF \( P_N(n) \).
(b) Find \( P_{N|B}(n) \), the conditional PMF of \( N \) given that there have been 20 consecutive tests without a failure.
(c) What is \( E[N|B] \), the expected number of tests given that there have been 20 consecutive tests without a failure?

7.2.8 \( W \) is the Gaussian \((0, 4)\) random variable. Given the event \( C = \{W > 0\} \), find the conditional PDF, \( f_{W|C}(w) \), the conditional expected value, \( E[W|C] \), and the conditional variance, \( \text{Var}[W|C] \).

7.2.9 The time between telephone calls at a telephone switch is the exponential random variable \( T \) with expected value 0.01.

(a) What is \( E[T|T > 0.02] \), the conditional expected value of \( T \)?
(b) What is \( \text{Var}[T|T > 0.02] \), the conditional variance of \( T \)?

7.2.10 As the final rider in the final 60 km time trial of the Tour de France, Roy must finish in time \( T \leq 1 \) hour to win the Tour. He has the choice of bike made of (1) carbon fiber or (2) titanium. On the carbon fiber bike, his speed \( V \) over the course is the continuous uniform random variable with \( E[V] = 58 \) km/hr and \( \text{Var}[V] = 12 \). On the titanium bike, \( V \) is the exponential random variable with \( E[V] = 60 \) km/hr.

(a) Roy chooses his bike to maximize \( P[W] \), the probability he wins the Tour. Which bike does Roy choose and what is \( P[W] \)?
(b) Suppose instead that Roy flips a fair
272  CHAPTER 7  CONDITIONAL PROBABILITY MODELS

7.2.11  For the distance $D$ of a shot-put toss in Problem 4.7.8, find the conditional PDFs $f_{D|D>0}(d)$ and $f_{D|D<0}(d)$.

7.3.1  $X$ and $Y$ are independent identical discrete uniform $(1, 10)$ random variables. Let $A$ denote the event that $\min(X, Y) > 5$. Find the conditional PMF $P_{X,Y|A}(x, y)$.

7.3.2  Continuing Problem 7.3.1, let $B$ denote the event that $\max(X, Y) \leq 5$. Find the conditional PMF $P_{X,Y|B}(x, y)$.

7.3.3  Random variables $X$ and $Y$ have joint PDF

$$f_{X,Y}(x, y) = \begin{cases} 6e^{-(2x+3y)} & x \geq 0, y \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Let $A$ be the event that $X + Y \leq 1$. Find the conditional PDF $f_{X,Y|A}(x, y)$.

7.3.4  $N$ and $K$ have joint PMF

$$P_{N,K}(n, k) = \begin{cases} \frac{(1-p)^{n-1}p}{n} & n=1,2,\ldots, \\ 0 & k=1,\ldots,n, \text{ otherwise.} \end{cases}$$

Let $B$ denote the event that $N \geq 10$.

(a) Find the conditional PMFs $P_{N|B}(n)$ and $P_{N,K|B}(n, k)$. Which should you find first?


7.3.5  $X$ and $Y$ have joint PDF

$$f_{X,Y}(x, y) = \begin{cases} (x+y)/3 & 0 \leq x \leq 1, \\ 0 & 0 \leq y \leq 2, \\ 0 & \text{otherwise.} \end{cases}$$

Let $A = \{Y \leq 1/2\}$.

(a) What is $P[A]$?

(b) Find $f_{X,Y|A}(x, y)$.

(c) Find $f_{X|A}(x)$ and $f_{Y|A}(y)$.

7.3.7  A study examined whether there was correlation between how much football a person watched and how bald the person was. The time $T$ watching football was measured on a 0, 1, 2 scale such that $T=0$ if a person never watched football, $T=1$ if a person watched football occasionally, and $T=2$ if a person watched a lot of football. Similarly, baldness $B$ was measured on the same scale: $B=0$ for a person with a full head of hair, $B=1$ for a person with thinning hair, and $B=2$ for a person who has not much hair at all. The experiment was to learn $B$ and $T$ for a randomly chosen person, equally likely to be a man (event $M$) or a woman (event $W$). The study found that given a person was a man (event $M$), random variables $B$ and $T$ were conditionally independent. Similarly, given that a person was a woman (event $W$), $B$ and $T$ were conditionally independent. Moreover, $B$ and $T$ had conditional joint PMFs

<table>
<thead>
<tr>
<th>$b$</th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P_{B</td>
<td>M}(b)$</td>
<td>0.2</td>
<td>0.3</td>
</tr>
<tr>
<td>$P_{T</td>
<td>M}(t)$</td>
<td>0.2</td>
<td>0.2</td>
</tr>
<tr>
<td>$P_{B</td>
<td>W}(b)$</td>
<td>0.6</td>
<td>0.3</td>
</tr>
<tr>
<td>$P_{T</td>
<td>W}(t)$</td>
<td>0.6</td>
<td>0.2</td>
</tr>
</tbody>
</table>

(a) Find the conditional PMF $P_{B,T|W}(b, t)$ of $B$ and $T$ given that a person is a woman.

(b) Find the conditional PMF $P_{B,T|M}(b, t)$ of $B$ and $T$ given that a person is a man.

(c) Find the joint PMF $P_{B,T}(b, t)$.

(d) Find the covariance of $B$ and $T$. Are $B$ and $T$ independent?
7.3.8 Random variables $X$ and $Y$ have joint PDF

$$f_{X,Y}(x,y) = \begin{cases} \frac{5x^2}{2} & -1 \leq x \leq 1; \\ 0 & 0 \leq y \leq x^2, \\ 0 & \text{otherwise}. \end{cases}$$

Let $A = \{Y \leq 1/4\}$.
(a) Find the conditional PDF $f_{X,Y|A}(x,y)$.
(b) Find $f_{Y|A}(y)$ and $E[Y|A]$.
(c) Find $f_{X|A}(x)$ and $E[X|A]$.

7.3.9 $X$ and $Y$ are independent random variables with PDFs

$$f_X(x) = \begin{cases} 2x & 0 \leq x \leq 1, \\ 0 & \text{otherwise}, \end{cases}$$

$$f_Y(y) = \begin{cases} 3y^2 & 0 \leq y \leq 1, \\ 0 & \text{otherwise}. \end{cases}$$

Let $A = \{X > Y\}$.
(a) What are $E[X]$ and $E[Y]$?
(b) What are $E[X|A]$ and $E[Y|A]$?

7.4.1 Given $X = x$,
- $Y_1$ is Gaussian with conditional expected value $x$ and conditional variance 1.
- $Y_2$ is Gaussian with conditional expected value $x$ and conditional variance $x^2$.

Find the conditional PDFs $f_{Y_1|X}(y_1|x)$ and $f_{Y_2|X}(y_2|x)$.

7.4.2 $X$ is the continuous uniform $(0,1)$ random variable. Given $X = x$, $Y$ has a continuous uniform $(0,x)$ PDF. What is the joint PDF $f_{X,Y}(x,y)$? Sketch the region of the $X,Y$ plane for which $f_{X,Y}(x,y) > 0$.

7.4.3 $X$ is the continuous uniform $(0,1)$ random variable. Given $X = x$, $Y$ is conditionally a continuous uniform $(0,1+x)$ random variable. What is the joint PDF $f_{X,Y}(x,y)$ of $X$ and $Y$?

7.4.4 $Z$ is a Gaussian $(0,1)$ noise random variable that is independent of $Y$ and $Y = X + Z$ is a noisy observation of $X$. What is the conditional PDF $f_{Y|X}(y|x)$?

7.4.5 A business trip is equally likely to take 2, 3, or 4 days. After a $d$-day trip, the change in the traveler's weight, measured as an integer number of pounds, is a uniform $(-d,d)$ random variable. For one such trip, denote the number of days by $D$ and the change in weight by $W$. Find the joint PMF $P_{D,W}(d,w)$.

7.4.6 $X$ and $Y$ have joint PDF

$$f_{X,Y}(x,y) = \begin{cases} \frac{(4x + 2y)}{3} & 0 \leq x \leq 1; \\ 0 & 0 \leq y \leq 1, \\ 0 & \text{otherwise}. \end{cases}$$

(a) For which values of $y$ is $f_{X|Y}(x|y)$ defined? What is $f_{X|Y}(x|y)$?
(b) For which values of $x$ is $f_{Y|X}(y|x)$ defined? What is $f_{Y|X}(y|x)$?

7.4.7 A student’s final exam grade depends on how close the student sits to the center of the classroom during lectures. If a student sits $r$ feet from the center of the room, the grade is a Gaussian random variable with expected value $80 - r$ and standard deviation $r$. If $r$ is a sample value of random variable $R$, and $X$ is the exam grade, what is $f_{X|R}(x|r)$?

7.4.8 $Y = ZX$ where $X$ is the Gaussian $(0,1)$ random variable and $Z$, independent of $X$, has PMF

$$P_Z(z) = \begin{cases} 1-p & z = -1, \\ p & z = 1. \end{cases}$$

True or False:
(a) $Y$ and $Z$ are independent.
(b) $Y$ and $X$ are independent.

7.4.9 At the One Top Pizza Shop, mushrooms are the only topping. Curiously, a pizza sold before noon has mushrooms with probability $p = 1/3$ while a pizza sold after noon never has mushrooms. Also, a pizza is equally likely to be sold before noon as
after noon. On a day in which 100 pizzas are sold, let \( N \) equal the number of pizzas sold before noon and let \( M \) equal the number of mushroom pizzas sold during the day. What is the joint PMF \( P_{M,N}(m,n) \)? Are \( M \) and \( N \) independent? Hint: Find the conditional PMF of \( M \) given \( N \).

7.4.10 \( ^2 \) Random variables \( X \) and \( Y \) have the joint PMF in the following table.

<table>
<thead>
<tr>
<th>( x )</th>
<th>( y = -1 )</th>
<th>( y = 0 )</th>
<th>( y = 1 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x = -1 )</td>
<td>3/16</td>
<td>1/16</td>
<td>0</td>
</tr>
<tr>
<td>( x = 0 )</td>
<td>1/6</td>
<td>1/6</td>
<td>1/6</td>
</tr>
<tr>
<td>( x = 1 )</td>
<td>0</td>
<td>1/8</td>
<td>1/8</td>
</tr>
</tbody>
</table>

(a) Are \( X \) and \( Y \) independent?

(b) The experiment from which \( X \) and \( Y \) are derived is performed sequentially. First, \( X \) is found, then \( Y \) is found. In this context, label the conditional branch probabilities of the following tree:

```
    ?
   / \  
   ?   ?
  /   /
 ?    ?
/     /
?     ?
/     /
?     ?
```

7.4.11 \( ^2 \) Flip a coin twice. On each flip, the probability of heads equals \( p \). Let \( X_i \) equal the number of heads (either 0 or 1) on flip \( i \). Let \( W = X_1 - X_2 \) and \( Y = X_1 + X_2 \). Find \( P_{W,Y}(w,y) \), \( P_{W|Y}(w|y) \), and \( P_{Y|W}(y|w) \).

7.4.12 \( ^2 \) Show that

\[
\lim_{\Delta \to 0} P \left[ x_1 < X \leq x_2, y < Y \leq y + \Delta \right] = \int_{x_1}^{x_2} \int_{y}^{y+\Delta} f_{X,Y}(x,y) \, dx.
\]

Hint: \( P[x_1 < X \leq x_2, y < Y \leq y + \Delta] \) can be written as an integral of \( f_{X,Y}(x,y) \).

7.4.13 \( ^\ast \) Packets arriving at an Internet router are either voice packets \((v)\) or data packets \((d)\). Each packet is a voice packet with probability \( p \), independent of any other packet. Observe packets at the Internet router until you see two voice packets. Let \( M \) equal the number of packets up to and including the first voice packet. Let \( N \) equal the number of packets observed up to and including the second voice packet. Find the conditional PMFs \( P_{M|N}(m|n) \) and \( P_{N|M}(n|m) \). Interpret your results.

7.4.14 \( ^\ast \) Suppose you arrive at a bus stop at time 0, and at the end of each minute, with probability \( p \), a bus arrives, or with probability \( 1 - p \), no bus arrives. Whenever a bus arrives, you board that bus with probability \( q \) and depart. Let \( T \) equal the number of minutes you stand at a bus stop. Let \( N \) be the number of buses that arrive while you wait at the bus stop.

(a) Identify the set of points \((n,t)\) for which \( P[N = n, T = t] > 0 \).

(b) Find \( P_{N,T}(n,t) \).

(c) Find the marginal PMFs \( P_{N}(n) \) and \( P_{T}(t) \).

(d) Find the conditional PMFs \( P_{N|T}(n|t) \) and \( P_{T|N}(t|n) \).

7.4.15 \( ^\ast \) Each millisecond at an Internet router, a packet independently arrives with probability \( p \). Each packet is either a data packet \((d)\) with probability \( q \) or a video packet \((v)\). Each data packet belongs to an email with probability \( r \). Let \( N \) equal the number of milliseconds required to observe the first 100 email packets. Let \( T \) equal the number of milliseconds you observe the router waiting for the first email packet. Find the marginal PMF \( P_{T}(t) \) and the conditional PMF \( P_{N|T}(n|t) \). Lastly, find the conditional PMF \( P_{T|N}(t|n) \).

7.5.1 \( ^\ast \) \( X \) and \( Y \) have joint PDF

\[
f_{X,Y}(x,y) = \begin{cases} 
2 & 0 \leq y \leq x \leq 1, \\
0 & \text{otherwise}.
\end{cases}
\]

Find the PDF \( f_{Y}(y) \), the conditional PDF \( f_{X|Y}(x|y) \), and the conditional expected value \( E[X|Y = y] \).
7.5.2 Let random variables $X$ and $Y$ have joint PDF $f_{X,Y}(x,y)$ given in Problem 7.5.1. Find the PDF $f_X(x)$, the conditional PDF $f_{Y|X}(y|x)$, and the conditional expected value $E[Y|X = x]$.

7.5.3 The probability model for random variable $A$ is

$$P_A(a) = \begin{cases} 1/3 & a = -1, \\ 2/3 & a = 1, \\ 0 & \text{otherwise.} \end{cases}$$

The conditional probability model for random variable $B$ given $A$ is:

$$P_{B|A}(b|a) = \begin{cases} 1/3 & b = 0, \\ 2/3 & b = 1, \\ 0 & \text{otherwise,} \end{cases}$$

$$P_{B|A}(b|1) = \begin{cases} 1/2 & b = 0, \\ 1/2 & b = 1, \\ 0 & \text{otherwise.} \end{cases}$$

(a) What is the probability model for random variables $A$ and $B$? Write the joint PMF $P_{A,B}(a,b)$ as a table.

(b) If $A = 1$, what is the conditional expected value $E[B|A = 1]$?

(c) If $B = 1$, what is the conditional PMF $P_{A|B}(a|1)$?

(d) If $B = 1$, what is the conditional variance $\text{Var}[A|B = 1]$ of $A$?

(e) What is the covariance $\text{Cov}[A, B]$?

7.5.4 For random variables $A$ and $B$ given in Problem 7.5.3, let $U = E[B|A]$. Find the PMF $P_U(u)$. What is $E[U] = E[E[B|A]]$?

7.5.5 Random variables $N$ and $K$ have the joint PMF:

$$P_{N,K}(n,k) = \begin{cases} 100^n e^{-100} \frac{1}{(n+1)^{k+1}} & n = 0, 1, \ldots, k = 0, 1, \ldots, n, \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find the marginal PMF $P_N(n)$ and the conditional PMF $P_{K|N}(k|n)$.

(b) Find the conditional expected value $E[K|N = n]$.

(c) Express the random variable $E[K|N]$ as a function of $N$ and use the iterated expectation to find $E[K]$.

7.5.6 Random variables $X$ and $Y$ have joint PDF

$$f_{X,Y}(x,y) = \begin{cases} 1/2 & -1 \leq x \leq y \leq 1, \\ 0 & \text{otherwise.} \end{cases}$$

(a) What is $f_Y(y)$?

(b) What is $f_{X|Y}(x|y)$?

(c) What is $E[X|Y = y]$?

7.5.7 Over the circle $x^2 + y^2 \leq r^2$, random variables $X$ and $Y$ have the uniform PDF

$$f_{X,Y}(x,y) = \begin{cases} 1/(\pi r^2) & x^2 + y^2 \leq r^2, \\ 0 & \text{otherwise.} \end{cases}$$

(a) What is $f_{Y|X}(y|x)$?

(b) What is $E[Y|X = x]$?

7.5.8 (Continuation of Problem 4.6.14) At time $t = 0$, the price of a stock is a constant $k$ dollars. At time $t > 0$ the price of a stock is a Gaussian random variable $X$ with $E[X] = k$ and $\text{Var}[X] = t$. At time $t$, a Call Option at Strike $k$ has value

$$V = (X - k)^+,$$

where the operator $(\cdot)^+$ is defined as $(x)^+ = \max(x, 0)$. Suppose that at the start of each $t = 30$ day month, you can buy the call option at strike $k$ at a price $D$ that is a random variable that fluctuates every month. You decide to buy the call only if the price $D$ is no more than a threshold $d^*$. What value of the threshold $d^*$ maximizes the expected return $E[R]$?

7.5.9 In a weekly lottery, each $1 ticket sold adds 50 cents to the jackpot that starts at $1 million before any tickets are sold. The jackpot is announced each morning to encourage people to play. On the morning of the $i$th day before the drawing, the current value of the jackpot $J_i$ is announced.
On that day, the number of tickets sold, \( N_t \), is a Poisson random variable with expected value \( J_t \). Thus, six days before the drawing, the morning jackpot starts at \$1 million and \( N_0 \) tickets are sold that day. On the day of the drawing, the announced jackpot is \( J_0 \) dollars and \( N_0 \) tickets are sold before the evening drawing. What are the expected value and variance of \( J \), the value of the jackpot the instant before the drawing? Hint: Use conditional expectations.

7.6.1 \( \triangleright \) You wish to measure random variable \( X \) with expected value \( E[X] = 1 \) and variance \( \text{Var}[X] = 1 \), but your measurement procedure yields the noisy observation \( Y = X + Z \), where \( Z \) is the Gaussian (0, 2) noise that is independent of \( X \).

(a) Find the conditional PDF \( f_{Z|X}(z|x) \) of \( Z \) given \( X = x \).

(b) Find the conditional PDF \( f_{Y|X}(y|x) \) of \( Y \) given \( X = x \). Hint: Given \( X = x \), \( Y = x + Z \).

7.6.2 \( \triangleright \) \( X \) and \( Y \) are jointly Gaussian random variables with \( E[X] = E[Y] = 0 \) and \( \text{Var}[X] = \text{Var}[Y] = 1 \). Furthermore, \( E[Y|X] = X/2 \). Find \( f_{X,Y}(x,y) \).

7.6.3 \( \triangleright \) A study of bicycle riders found that a male cyclist’s speed \( X \) (in miles per hour over a 100-mile “century” ride) and weight \( Y \) (kg) could be modeled by a bivariate Gaussian PDF \( f_{X,Y}(x,y) \) with parameters \( \mu_X = 20, \sigma_X = 2, \mu_Y = 75, \sigma_Y = 5 \) and \( \rho_{X,Y} = -0.6 \). In addition, a female cyclist’s speed \( X’ \) and weight \( Y’ \) could be modeled by a bivariate Gaussian PDF \( f_{X’,Y’}(x’,y’) \) with parameters \( \mu_{X’} = 15, \sigma_{X’} = 2, \mu_{Y’} = 50, \sigma_{Y’} = 5 \) and \( \rho_{X’,Y’} = -0.6 \). For men and women, the negative correlation of speed and weight reflects the common wisdom that fast cyclists are thin. As it happens, cycling is much more popular among men than women; in a mixed group of cyclists, a cyclist is a male with probability \( p = 0.80 \).

You suspect it’s OK to ignore the differences between men and women since for both groups, weight and speed are negatively correlated, with \( \rho = -0.6 \). To convince yourself this is OK, you decide to study the speed \( X \) and weight \( Y \) of a cyclist randomly chosen from a large mixed group of male and female cyclists. How are \( X \) and \( Y \) correlated? Explain your answer.

7.6.4 \( \triangleright \) Let \( X_1 \) and \( X_2 \) have a bivariate Gaussian PDF with correlation coefficient \( \rho_{12} \) such that each \( X_i \) is a Gaussian \((\mu_i, \sigma_i)\) random variable. Show that \( Y = X_1 X_2 \) has variance

\[
\text{Var}[Y] = \sigma_1^2 \sigma_2^2 (1 + \rho_{12}^2) + \sigma_1^2 \mu_1^2 + \mu_1 \sigma_2^2 - \mu_1^2 \mu_2^2.
\]

Hints: Look ahead to Problem 9.24 and also use the iterated expectation to find

\[
E[X_1^2 X_2^2] = E[E[X_1^2 X_2^2|X_2]].
\]

7.6.5 \( \triangleright \) Use the iterated expectation for a proof of Theorem 5.19 without integrals.

7.7.1 \( \triangleright \) For the modem receiver voltage \( X \) with PDF given in Example 7.8, use MATLAB to plot the PDF and CDF of random variable \( X \). Write a MATLAB function \( x=modemrv(m) \) that produces \( m \) samples of the modem voltage \( X \).

7.7.2 \( \triangleright \) For the quantizer of Example 7.6, we showed in Problem 7.1.9 that the quantization noise \( Z \) is nonuniform if \( X \) is nonuniform. In this problem, we examine whether it is a reasonable approximation to model the quantization noise as uniform. Consider the special case of a Gaussian \((0, 1)\) random variable \( X \) passed through a uniform \( b \)-bit quantizer over the interval \((-r/2, r/2)\) with \( r = 6 \). Does a uniform approximation get better or worse as \( b \) increases? Write a MATLAB program to generate histograms for \( Z \) to answer this question.
In this chapter, we expand on the concepts presented in Chapter 5. While Chapter 5 introduced the CDF and PDF of $n$ random variables $X_1, \ldots, X_n$, this chapter focuses on the random vector $\mathbf{X} = [X_1 \quad \ldots \quad X_n]'$. A random vector treats a collection of $n$ random variables as a single entity. Thus, vector notation provides a concise representation of relationships that would otherwise be extremely difficult to represent.

The first section of this chapter presents vector notation for a set of random variables and the associated probability functions. The subsequent sections define marginal probability functions of subsets of $n$ random variables, $n$ independent random variables, independent random vectors, and expected values of functions of $n$ random variables. We then introduce the covariance matrix and correlation matrix, two collections of expected values that play an important role in stochastic processes and in estimation of random variables. The final two sections cover Gaussian random vectors and the application of MATLAB, which is especially useful in working with multiple random variables.

8.1 Vector Notation

A random vector with $n$ dimensions is a concise representation of a set of $n$ random variables. There is a corresponding notation for the probability model (CDF, PMF, or PDF) of a random vector.

When an experiment produces two or more random variables, vector and matrix notation provide a concise representation of probability models and their properties. This section presents a set of definitions that establish the mathematical notation of random vectors. We use boldface notation $\mathbf{x}$ for a column vector. Row vectors are transposed column vectors; $\mathbf{x}'$ is a row vector. The components of a column vector are, by definition, written in a column. However, to save space, we will often